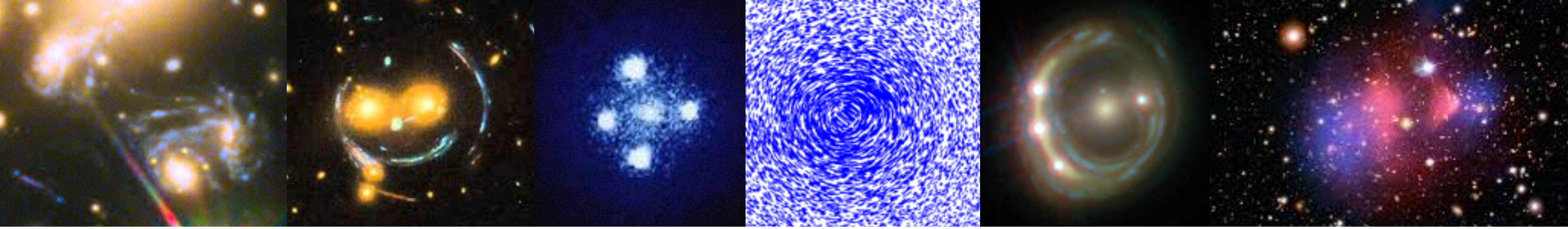

PHY-765 SS19 Gravitational Lensing Week 5

Multiple Images

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Last week - what did we learn?

- We derived the lens equation:

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \boldsymbol{\alpha}(\boldsymbol{\theta})$$

- A source with true position **β** on the sky can be seen by an observer to be located at angular position **θ** under the deflection **α**(**θ**).

- And defined (for the point mass) the

Critical Surface Mass Density

$$\Sigma_{\rm cr} \equiv \frac{c^2}{4\pi G}\frac{D_{\rm S}}{D_{\rm L}D_{\rm LS}}$$

Convergence

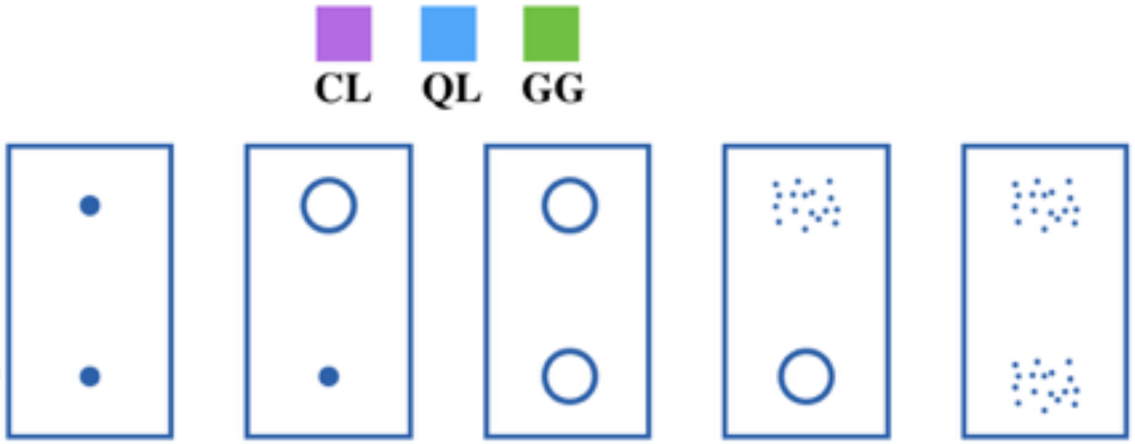
$$\kappa(\boldsymbol{\theta}) \equiv \frac{\Sigma(D_{\rm L}\boldsymbol{\theta})}{\Sigma_{\rm cr}}$$

Einstein Radius

$$\theta_E \equiv \sqrt{\frac{4MG}{c^2}\frac{D_{\rm LS}}{D_{\rm S}D_{\rm L}}}$$

The aim of today

- Explore the first consequence of the lens equation: **multiple images**
- Describe this for a few simplistic lens models
- Introduce the concepts of critical curves and caustics
- SN Refsdal - a spectacular example of multiple images



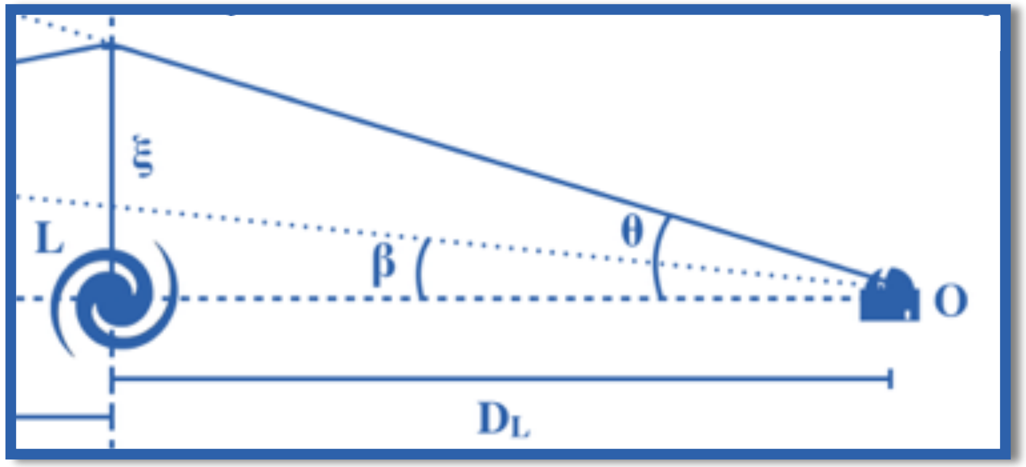
Multiple Images from the Point Mass Lens

- Last week we described the point mass lens:

$$\theta_E \equiv \sqrt{\frac{4MG}{c^2}\frac{D_{\rm LS}}{D_{\rm S}D_{\rm L}}} \qquad \boldsymbol{\alpha}(\boldsymbol{\theta}) = \frac{4MG}{c^2}\frac{D_{\rm LS}}{D_{\rm S}D_{\rm L}}\frac{\boldsymbol{\theta}}{|\boldsymbol{\theta}|^2} = \frac{\theta_{\rm E}^2}{|\boldsymbol{\theta}|^2}\boldsymbol{\theta}$$

- So we can write the lens equation as:

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \frac{\theta_{\rm E}^2}{|\boldsymbol{\theta}|^2}\boldsymbol{\theta}$$



- The Einstein radius was defined at $\boldsymbol{\beta} = 0$
  - The lens equation is solved for all $\theta = \theta_E$
- If imperfect alignment then x and y components of the lens equation are:

$$\beta = \theta_x\left[1 - \frac{\theta_{\rm E}^2}{\theta^2}\right] \qquad 0 = \theta_y\left[1 - \frac{\theta_{\rm E}^2}{\theta^2}\right]$$

  - Assuming coordinate system aligned such that $\boldsymbol{\beta} = \beta\hat{x}$ and $\beta > 0$

Multiple Images from the Point Mass Lens

- If $\theta_y \neq 0$, then we would have $\theta^2 \equiv \theta_x^2 + \theta_y^2 = \theta_E^2$

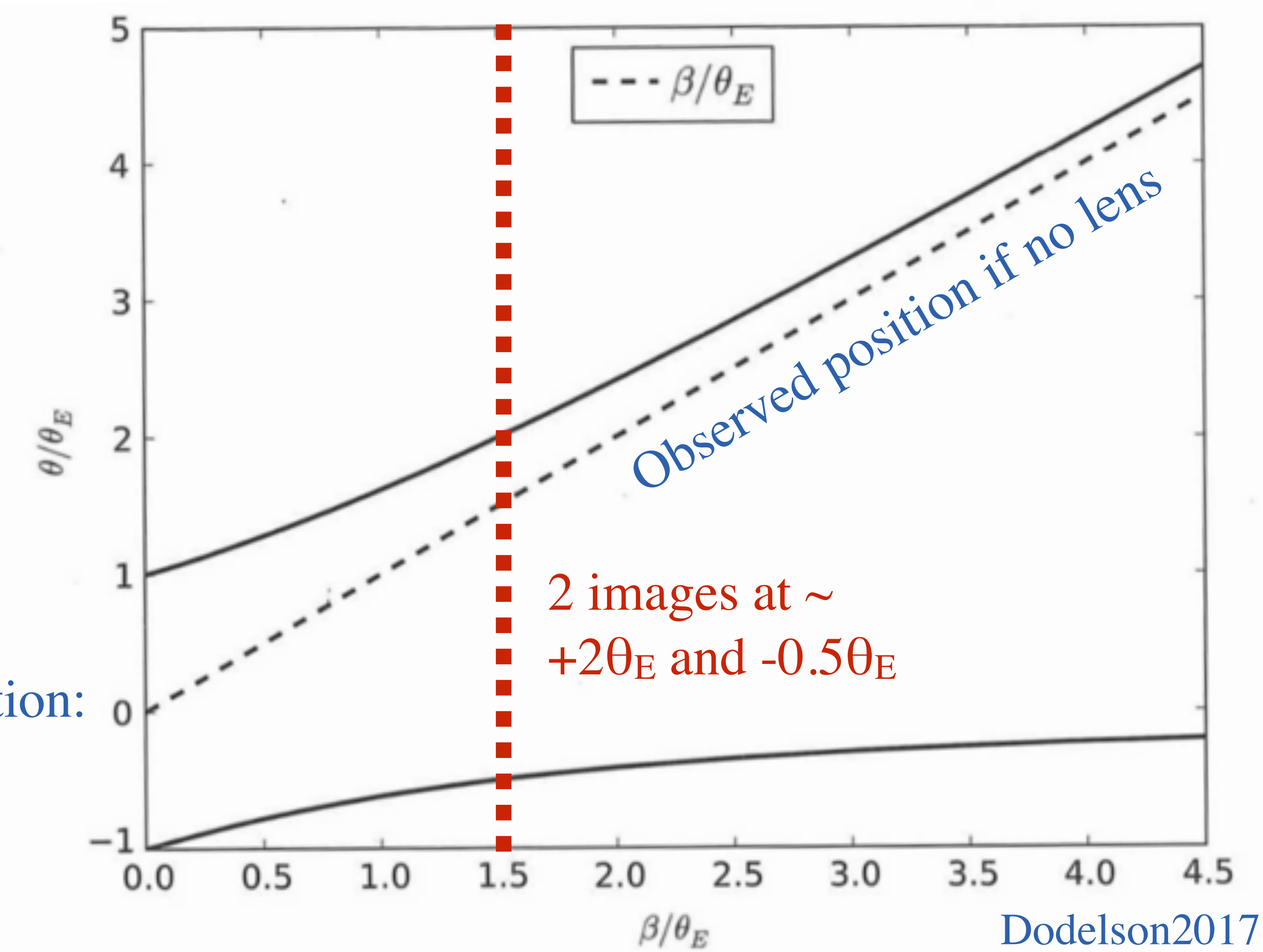
$$0 = \theta_y \left[1 - \frac{\theta_E^2}{\theta^2}\right]$$

$$\beta = \theta_x \left[1 - \frac{\theta_E^2}{\theta^2}\right]$$

- But then $\beta = 0$ is violating $\beta > 0$
- So we must conclude that $\theta_y = 0$ (for the simple point mass lens)
  - I.e. the lens equation mapping is determined solely by the x-component
- Hence,

$$\theta_\pm = \frac{\beta}{2}\left[1 \pm \sqrt{1 + \frac{4\theta_E^2}{\beta^2}}\right]$$

Lens position:
θ = 0



Dodelson2017

Multiple Images from the Point Mass Lens

- The limits for this setup are therefore:

$$\theta_\pm \simeq \pm\theta_{\rm E} + \frac{\beta}{2} \qquad (\beta \ll \theta_{\rm E})$$

$$\theta_+ \simeq \beta + \frac{\theta_{\rm E}^2}{\beta} \quad \& \quad \theta_- \simeq -\frac{\theta_{\rm E}^2}{\beta} \qquad (\beta \gg \theta_{\rm E})$$

$$\theta_\pm = \frac{\beta}{2}\left[1 \pm \sqrt{1 + \frac{4\theta_{\rm E}^2}{\beta^2}}\right]$$

  - Using the Taylor expansion $\sqrt{1+\epsilon} \simeq 1 + \frac{1}{2}\epsilon - \ldots$ for the limit $\beta \gg \theta_{\rm E}$

Spherically Symmetric Mass Distribution

- To start generalizing these ideas we first look at the spherical distribution
- For a spherical distribution the convergence is independent of the direction

$$\kappa(\boldsymbol{\theta}) \equiv \frac{\Sigma(D_{\rm L}\boldsymbol{\theta})}{\Sigma_{\rm cr}} \qquad \rightarrow \qquad \kappa(\theta) \equiv \frac{\Sigma(D_{\rm L}\theta)}{\Sigma_{\rm cr}}$$

- Such that the deflection angle becomes

$$\boldsymbol{\alpha}(\boldsymbol{\theta}) = \frac{1}{\pi}\int d^2\theta' \kappa(\theta') \frac{\boldsymbol{\theta}-\boldsymbol{\theta}'}{|\boldsymbol{\theta}-\boldsymbol{\theta}'|^2}$$

(from week 3)

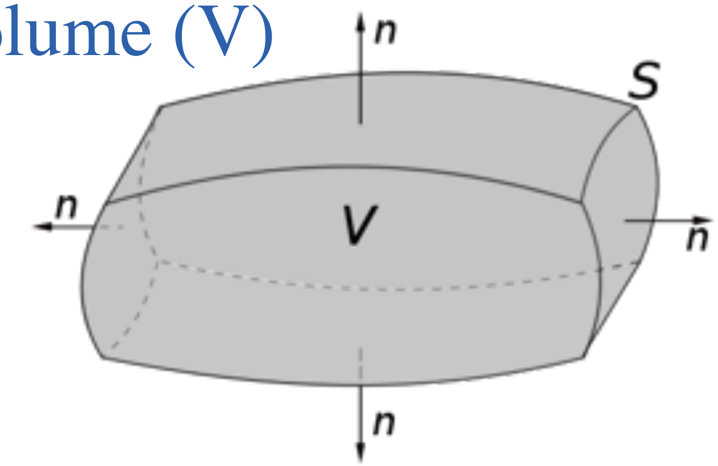
- **α**(**θ**) is a vector. Only relevant vector is **θ** (κ doesn't care) so we can write:

$$\boldsymbol{\alpha}(\boldsymbol{\theta}) = A(\theta)\,\boldsymbol{\theta}$$

- The goal is now to determine the coefficient *A(θ)*
- Remember the divergence (Gauss') theorem:

$$\iiint_{\rm V} \boldsymbol{\nabla}\cdot\boldsymbol{F}\,dV = \iint_{\rm S} \boldsymbol{F}\cdot\boldsymbol{n}\,dS$$

Surface (S) enclosing volume (V)



Spherically Symmetric Mass Distribution

- Using the planar version of the divergence theorem we get:

$$\int_{\theta' < \theta_{\max}} d^2\theta \, \boldsymbol{\nabla} \cdot \boldsymbol{\alpha}(\boldsymbol{\theta}) = \oint_{\mathrm{C}} d\phi \, \boldsymbol{\theta} \cdot \boldsymbol{\alpha}$$

Disk with radius θ_{max} Circumference of disk

- Right-hand side integrand is just $A(\theta_{\max})\theta_{\max}^2$ (independent of ϕ)
- Left-hand side: $$\boldsymbol{\nabla} \cdot \boldsymbol{\alpha}(\boldsymbol{\theta}) = \frac{1}{\pi} \int d^2\theta' \kappa(\theta') \, \boldsymbol{\nabla}_{\boldsymbol{\theta}} \cdot \frac{\boldsymbol{\theta} - \boldsymbol{\theta}'}{|\boldsymbol{\theta} - \boldsymbol{\theta}'|^2}$$
  - The relation: $\nabla \ln |\boldsymbol{\theta}| = \boldsymbol{\theta}/|\boldsymbol{\theta}|^2$ (Exercise 3.3)
  - The identity: $\nabla^2 \ln |\boldsymbol{\theta}| = 2\pi \delta_{\mathrm{D}}^2(\boldsymbol{\theta})$
- Gives us that:

$$\boldsymbol{\nabla} \cdot \boldsymbol{\alpha}(\boldsymbol{\theta}) = 2\kappa(\theta)$$

- Hence inserting shows that

$$\int_{\theta' < \theta_{\max}} d^2\theta \; 2\kappa(\theta) = 2\pi \; A(\theta_{\max})\theta_{\max}^2$$

Spherically Symmetric Mass Distribution

- such that $$A(\theta) = \langle \kappa(\theta) \rangle$$
- If we define the mean normalized surface mass density as:

$$\langle \kappa(\theta) \rangle \equiv \frac{1}{\pi \theta^2} \int_{\theta' < \theta} d^2\theta' \kappa(\theta')$$

- So we can express the lens equation for the spherical symmetric mass as

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \langle \kappa(\theta) \rangle \boldsymbol{\theta}$$

- As κ is the ratio between surface density at angular distance θ from the lens (normalized by the critical surface density) this dictates that:

  The deflection (**β** - **θ**) a distance θ from the lens is governed by the mass contained within the cylinder of radius $\xi = D_L \theta$.

- which gives that

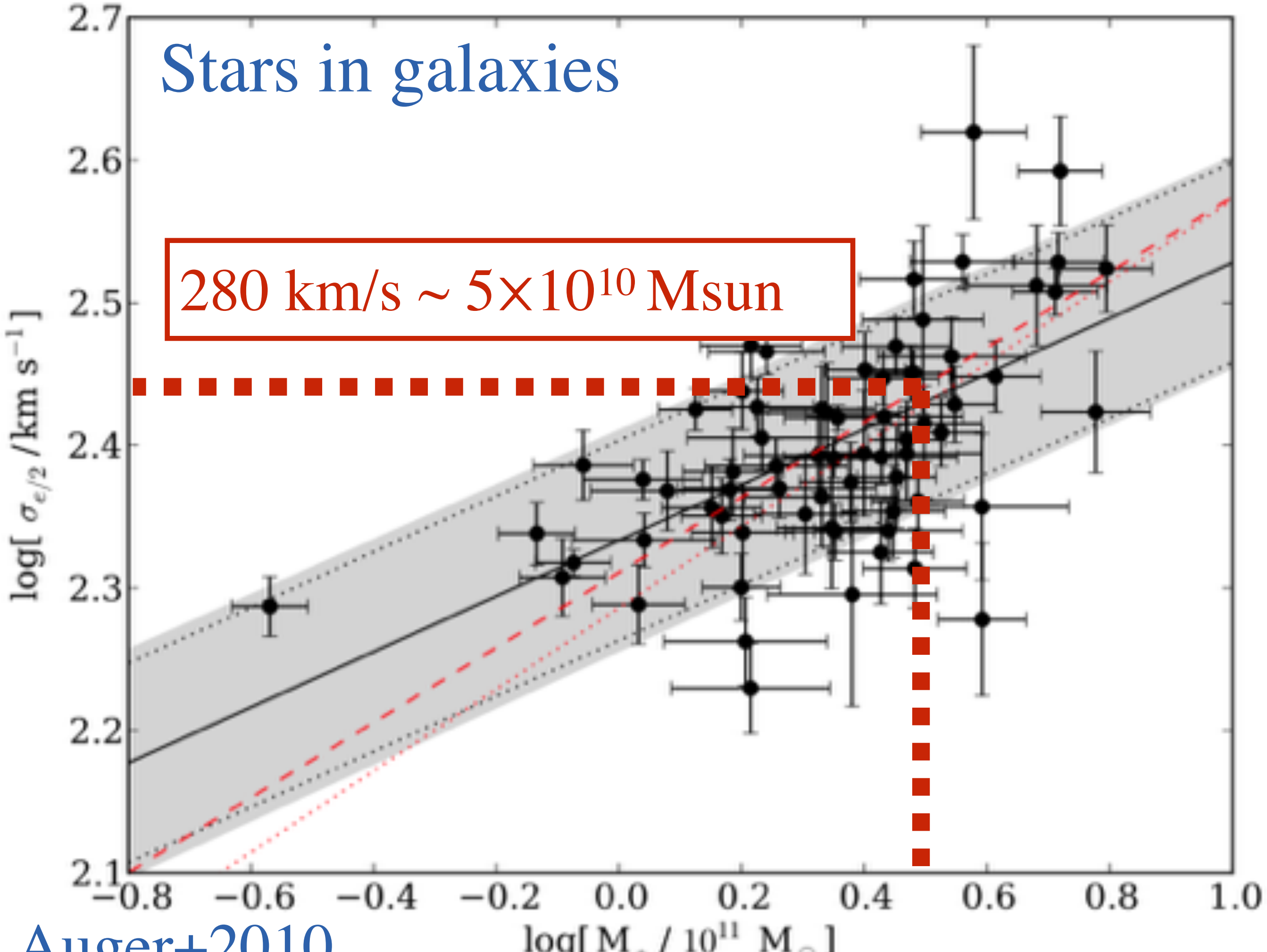
$$\langle \kappa(\theta) \rangle = \frac{M(R = D_L \theta)}{\pi D_L^2 \theta^2 \sum_{cr}}$$

The Isothermal Sphere (IS)

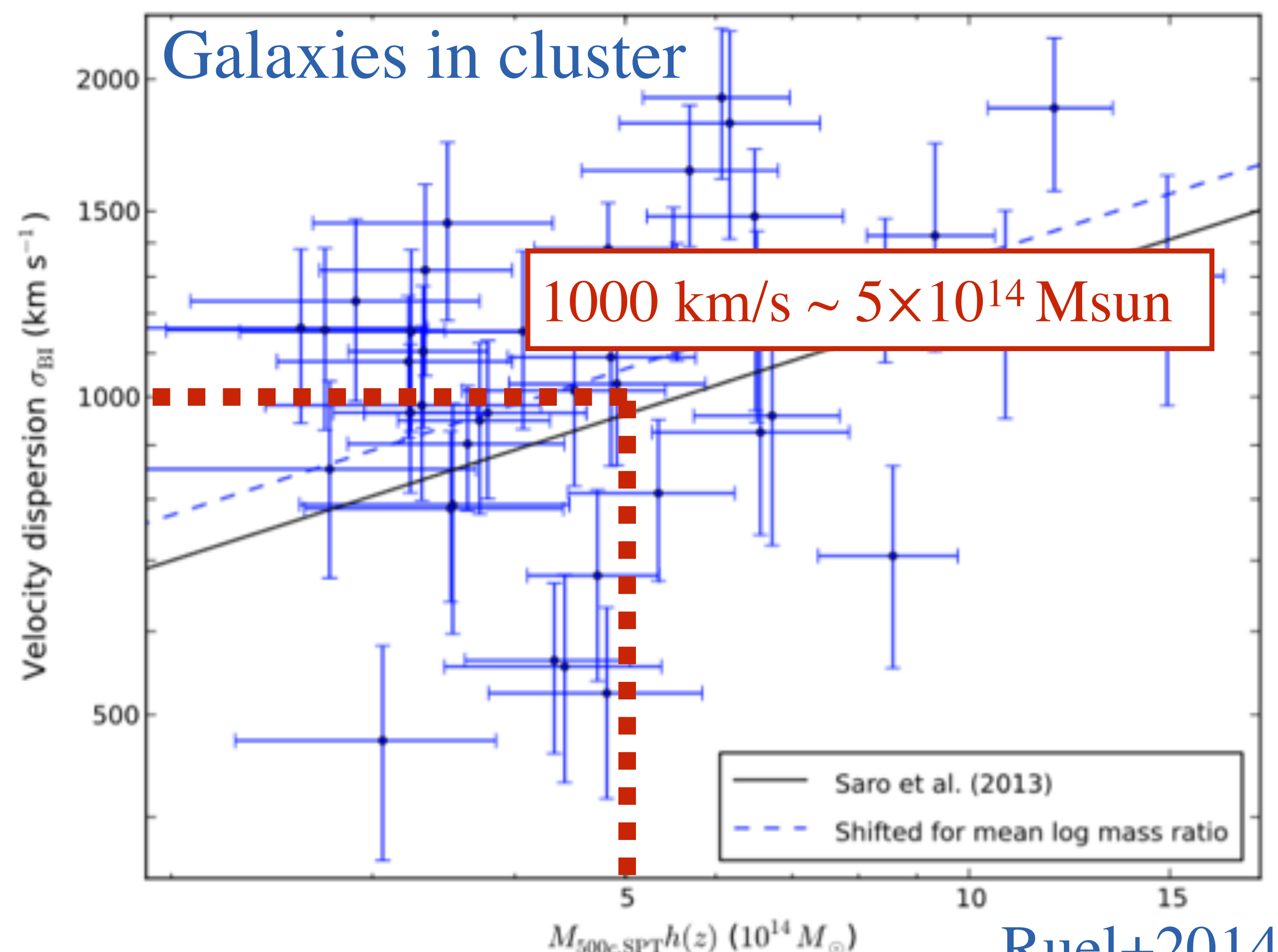
- The result of the spherical symmetric mass distribution can be applied to

$$\rho(r) = \frac{\sigma^2}{2\pi G(r^2 + r_{\rm core}^2)}$$

  - $r_{\rm core}$ is the core radius (the density profile turns flat in the core)
  - σ is the velocity dispersion of the lens



Auger+2010



Ruel+2014

The Isothermal Sphere (IS)

- Using this density profile we have the surface density:

$$\Sigma(R) = \frac{\sigma^2}{2\pi G}\int_{-\infty}^{+\infty}\frac{dz}{R^2+z^2+r_{\rm core}^2} \quad \rightarrow \quad \Sigma(R) = \frac{\sigma^2}{2G\sqrt{R^2+r_{\rm core}^2}}$$

- Which can be used to express the average surface density within a radius R

$$M(R) = 2\pi\int_0^R dR' R'\Sigma(R') \quad \rightarrow \quad M(R) = \frac{\pi\sigma^2}{G}\left[\sqrt{R^2+r_{\rm core}^2} - r_{\rm core}\right]$$

- Using $\theta = R/D_{\rm L}$ and $\theta_{\rm core} \equiv r_{\rm core}/D_{\rm L}$ we get the lens equation

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \frac{\theta_0}{\theta^2}\left[\sqrt{\theta^2+\theta_{\rm core}^2} - \theta_{\rm core}\right]\boldsymbol{\theta}$$

  - by defining the IS Einstein radius as:

$$\theta_0 \equiv \frac{4\pi\sigma^2 D_{\rm LS}}{D_{\rm S}c^2}$$

(Exercise 5.3)

The Singular Isothermal Sphere (SIS)

- For an isothermal sphere with no core ($\theta_{\rm core} = 0$) the lens equation becomes

$$\boldsymbol{\beta} = \boldsymbol{\theta}\left[1 - \frac{\theta_0}{|\boldsymbol{\theta}|}\right]$$

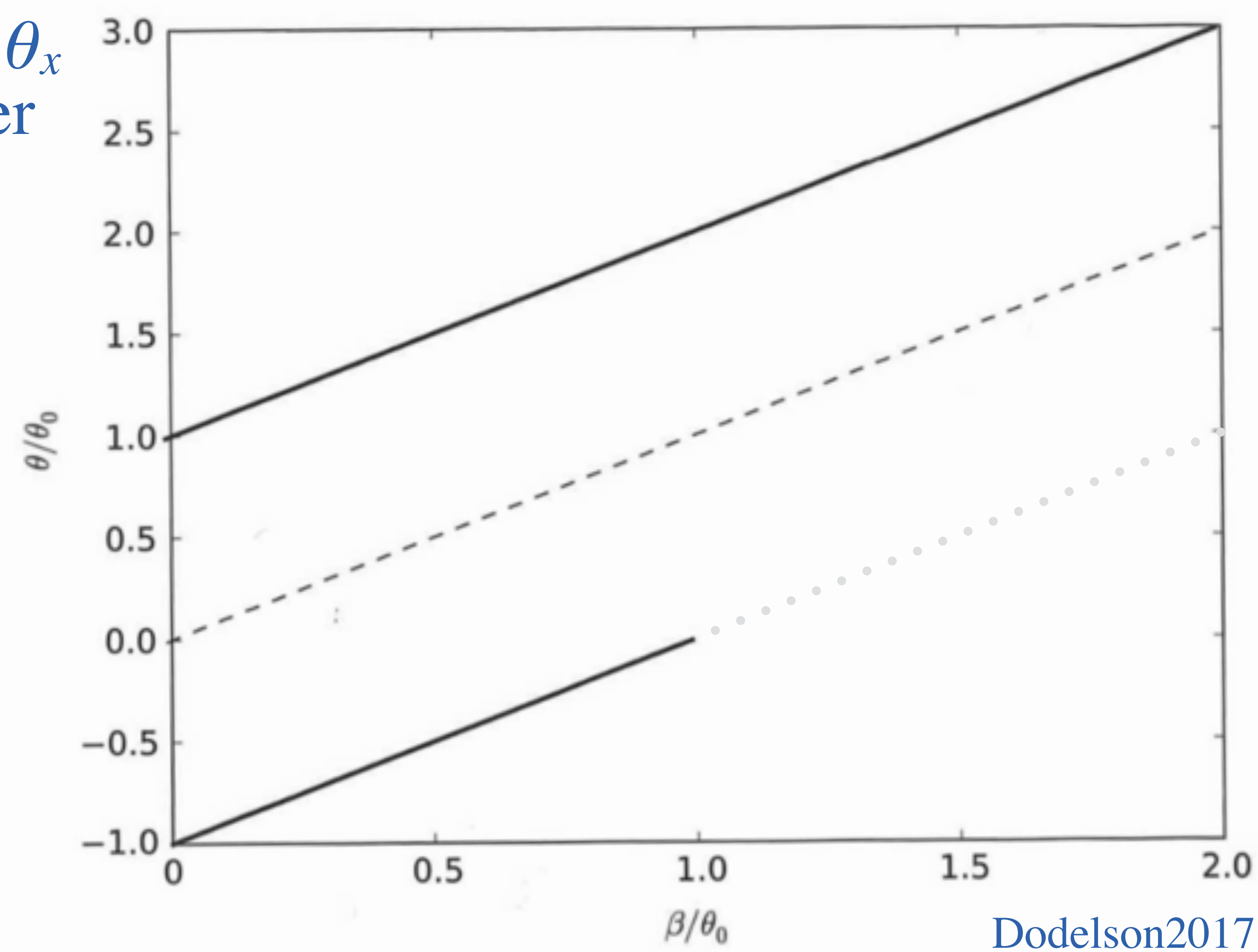
  - where $\beta = 0$ generates an image (Einstein) ring motivating the θ_0 definition

- Like for the point mass only the θ_x component is relevant to consider

$$\theta_+ = \beta + \theta_0$$

$$\theta_- = \beta - \theta_0$$

(but only for $\beta < \theta_0$)



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Cored Isothermal Sphere (CIS)

- The mass density of real galaxies does not rise all the way into the center
- So even though SIS is simple, the assumption that $r_{\rm core} = 0$ is poor.
- Using the definition of θ_0 and solving the lens equation for $\beta = 0$ we have

$$\theta^2 = \theta_0 \left[\sqrt{\theta^2 + \theta_{\rm core}^2} - \theta_{\rm core}\right] \qquad \theta_{\rm E} = \theta_0 \sqrt{1 - 2\frac{\theta_{\rm core}}{\theta_0}}$$

- Hence, the size of the core determines when an Einstein ring can exist
  - I.e., if $\theta_{\rm core} > 0.5\theta_0$ then an Einstein ring cannot be formed
- Happens at ~1" for lens at 1Gpc (Exercise 5.3)
  - Actual physical size differes as θ_0 depends on $z_{\rm L}, z_{\rm S}$ and σ
- If the lens-source alignment is not perfect the lens equation becomes

$$\theta(\beta - \theta) = -\theta_0 \left[\sqrt{\theta^2 + \theta_{\rm core}^2} - \theta_{\rm core}\right]$$

  - Rather complex to solve for individual images.

Cored Isothermal Sphere (CIS)

- But plotting β as a function of θ and flipping the axes is easy:

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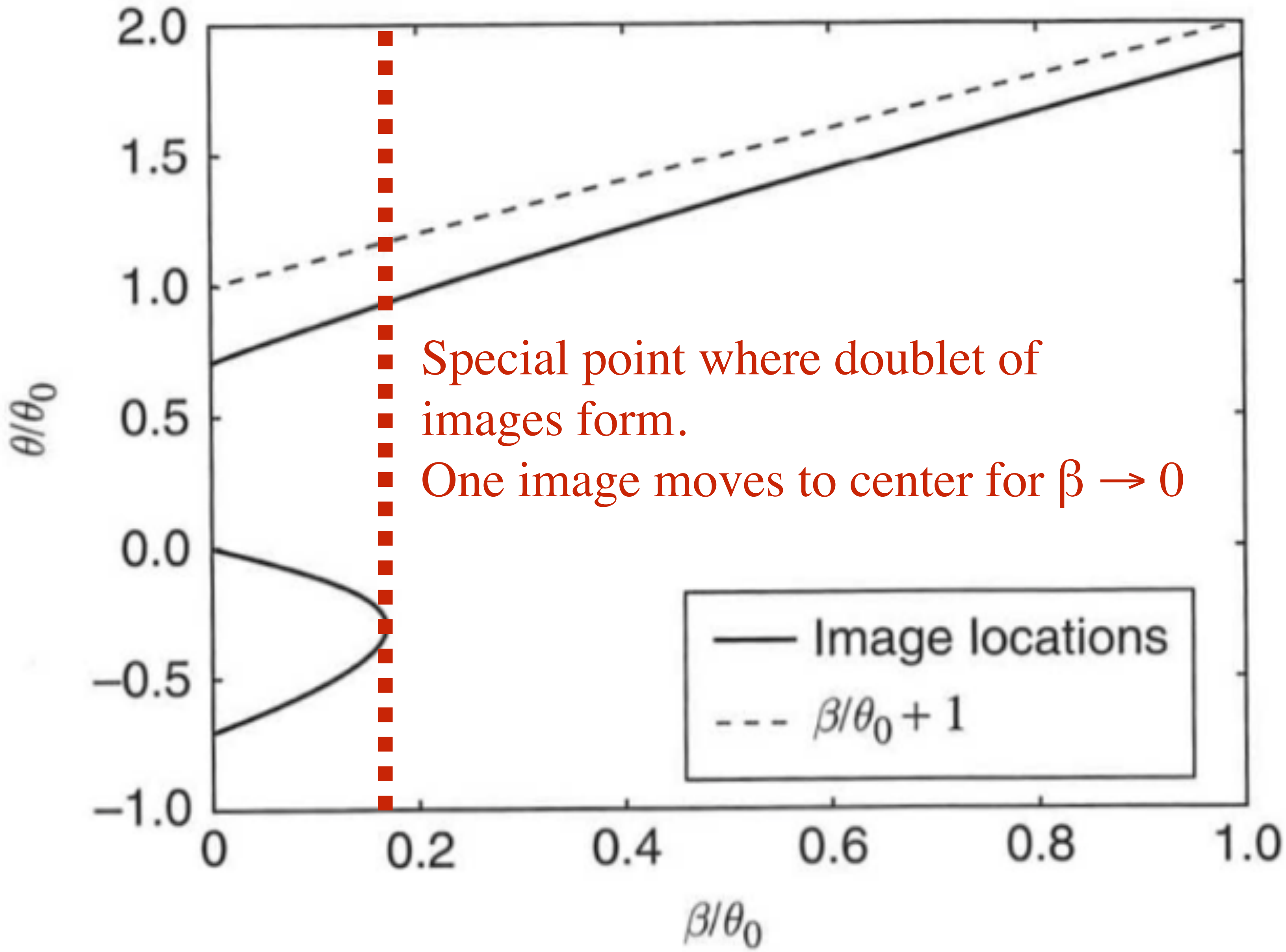

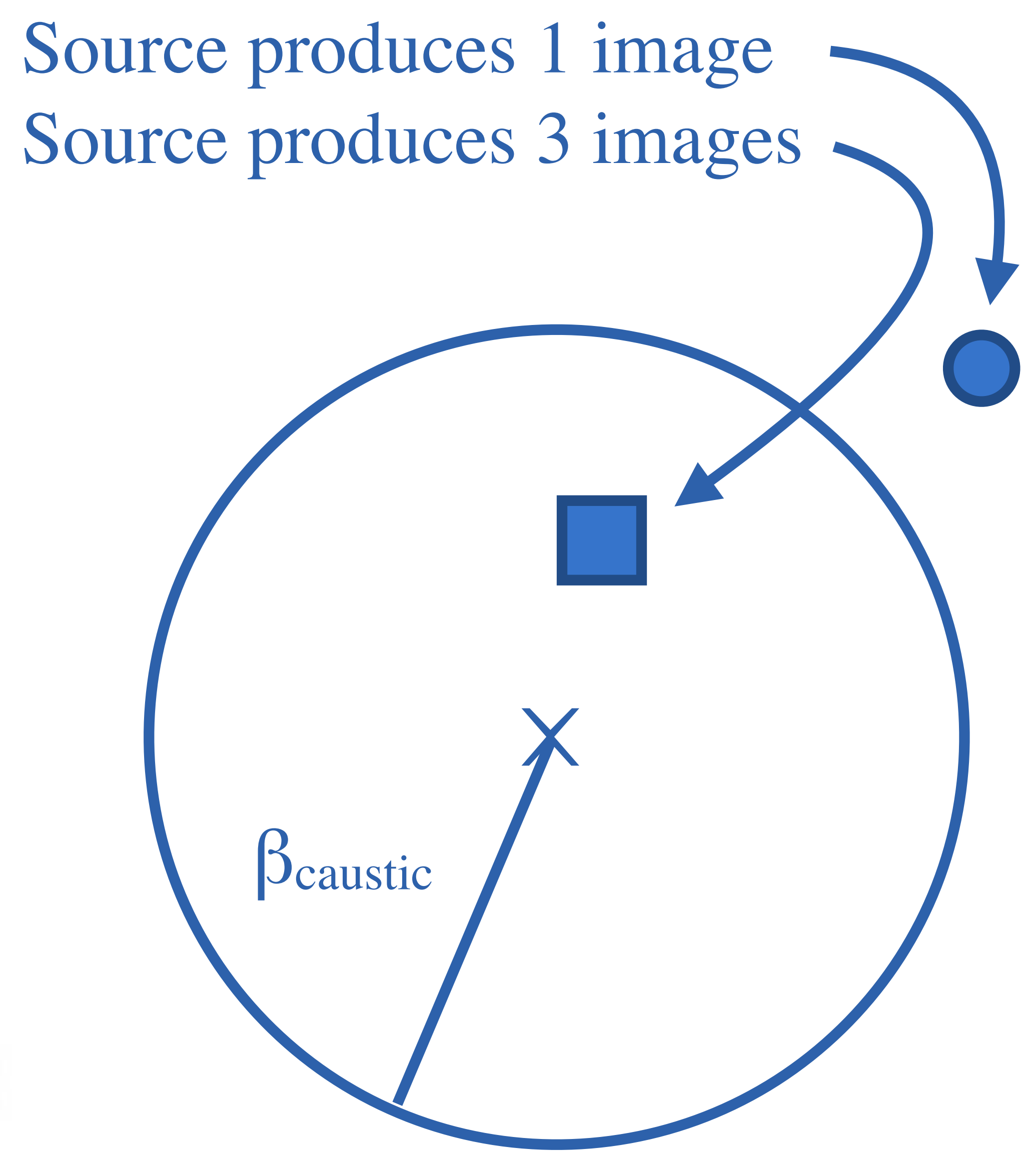

- In 1D this special value $\beta_{caustic}$ is on a curve, but more generally it defines a circle of radius $\beta_{caustic}$ in the source plane

Caustics

- We want to determine the source position β_{caustic}
- Returning to the lens eq. on the form $\boldsymbol{\beta} = \boldsymbol{\theta} - \frac{\theta_0}{\theta^2}\left[\sqrt{\theta^2 + \theta_{\text{core}}^2} - \theta_{\text{core}}\right]\boldsymbol{\theta}$
- $\beta \to \pm\infty$ for $\theta \to \pm\infty$ so for three solutions there must be 2 extrema
- All values of β below $\beta(\theta_-) \equiv \beta_{\text{caustic}}$ produces 3 images (where $\beta > 0$)
- Finding the extrema is done by obtaining the solutions to: $\frac{d\beta}{d\theta} = 0$

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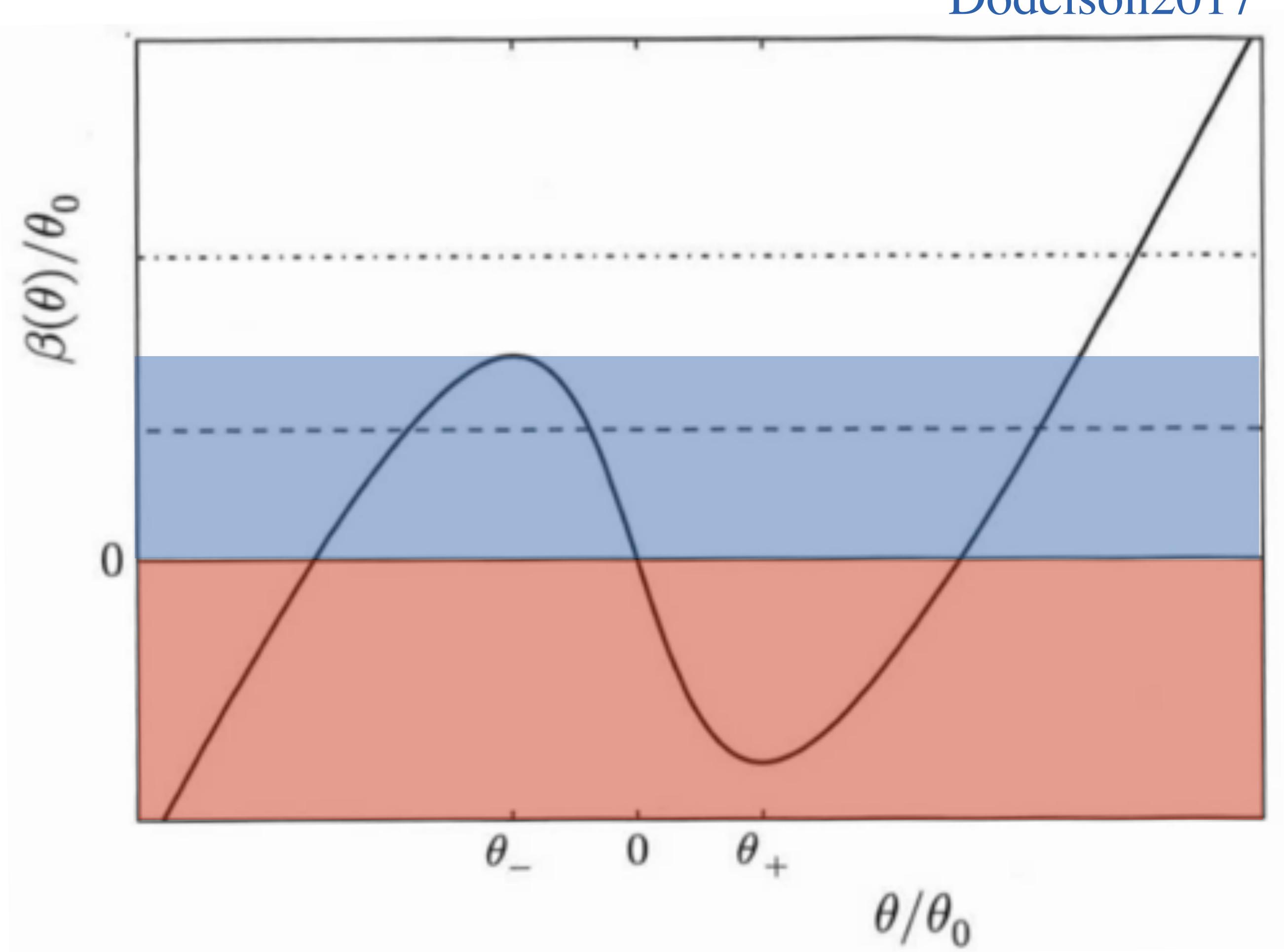
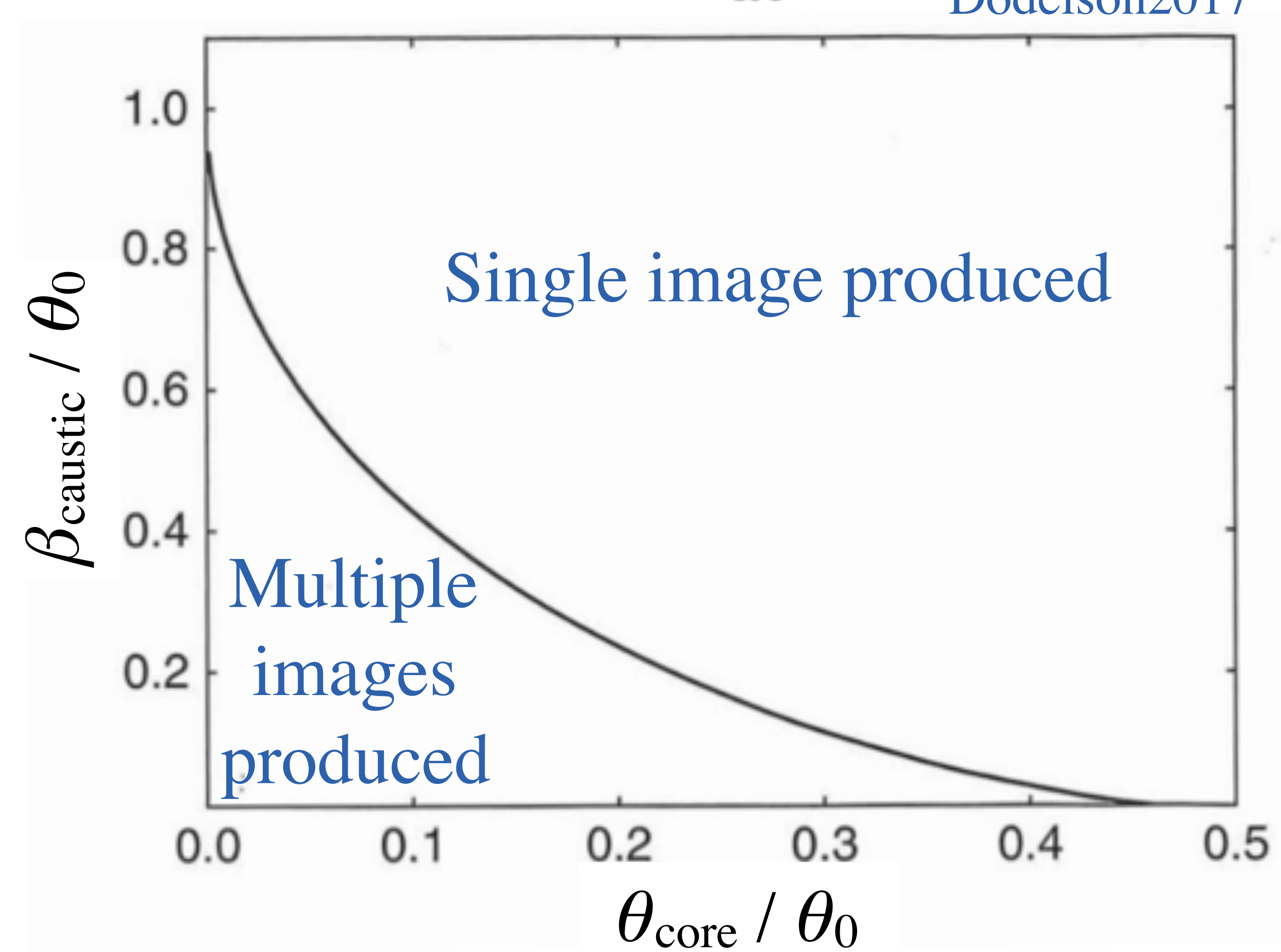
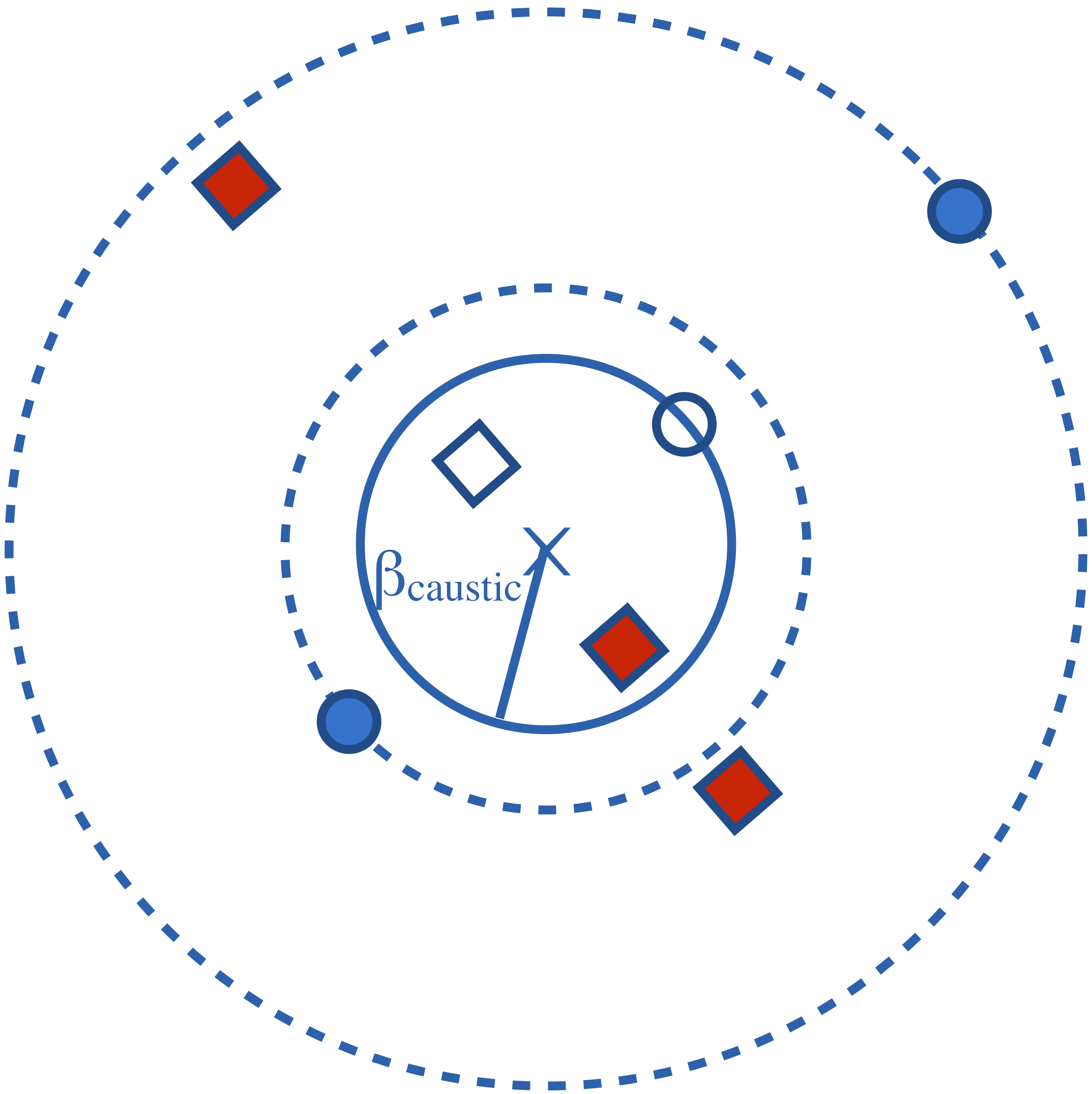
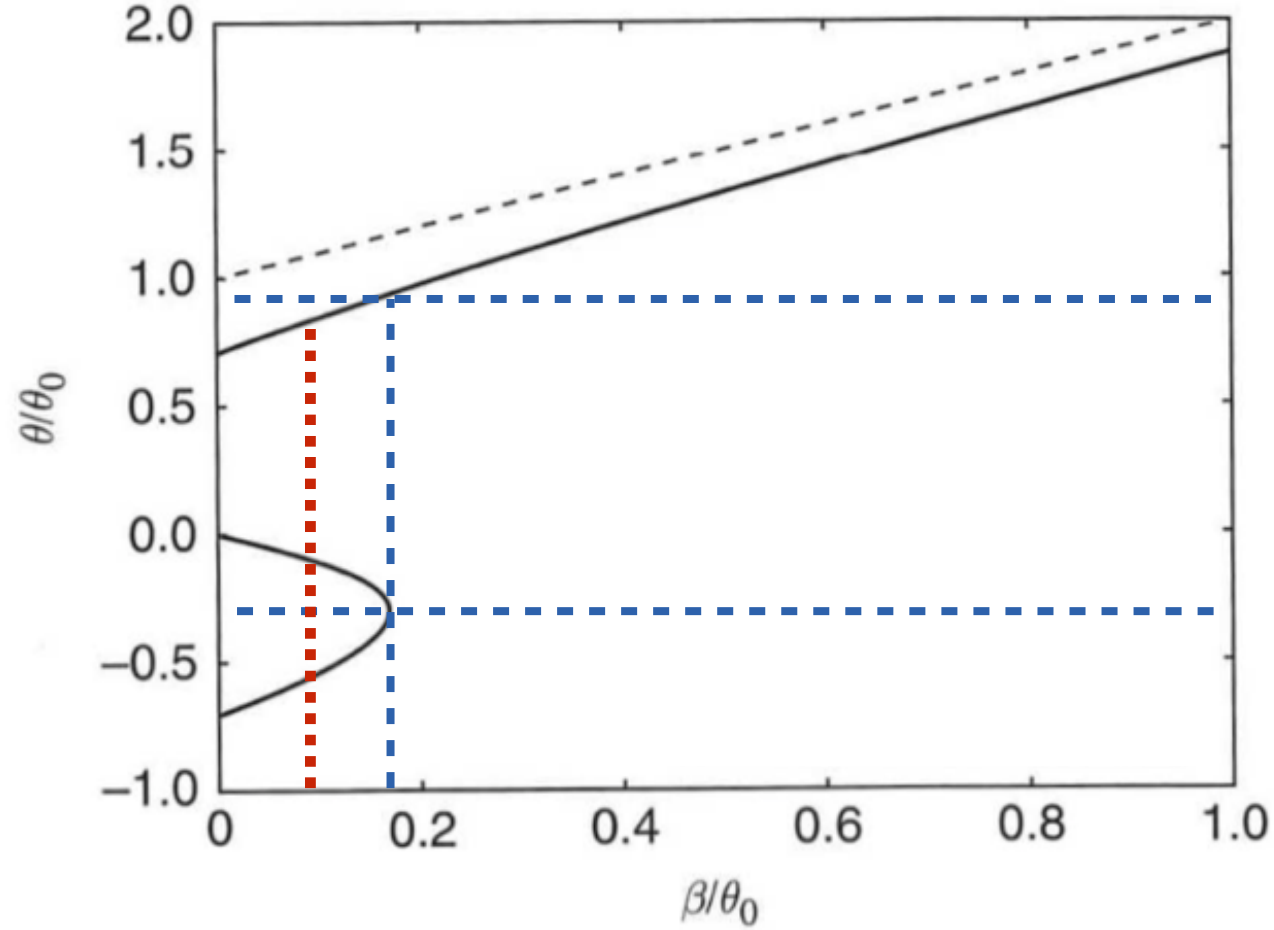


Dodelson2017



Critical Curves



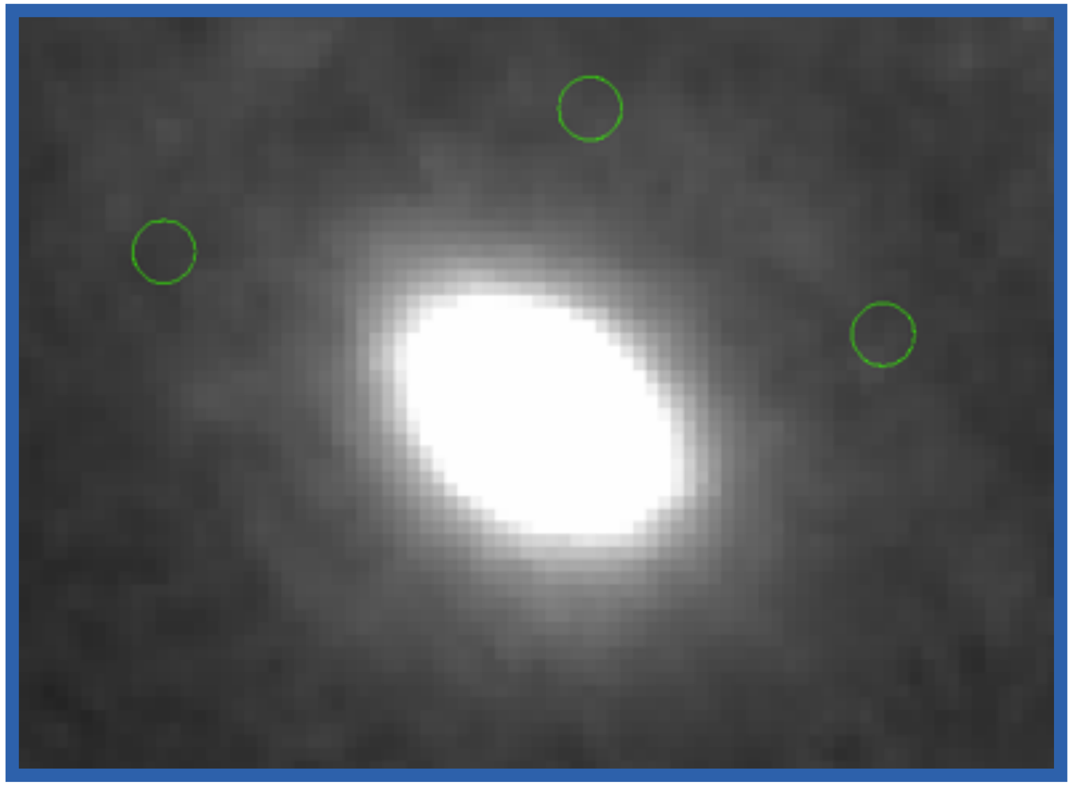


- The critical curves are define as the curves in the *lens* (image) plane where the images fall if the source is on the caustic in the source plane.

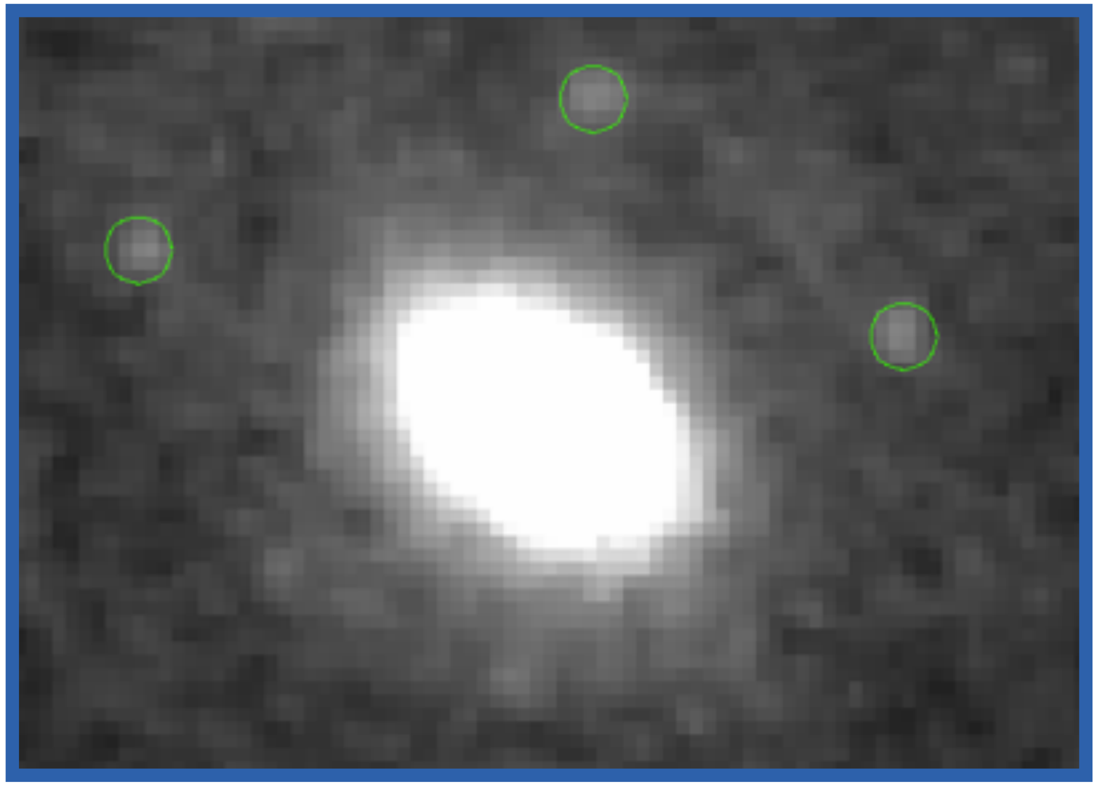
- Will return to caustic and critical curves when talking about magnification…

SN Refsdal

- Nov 2014: Discovered in MACS1149 data from the GLASS program



Existing Imaging



GLASS F104W

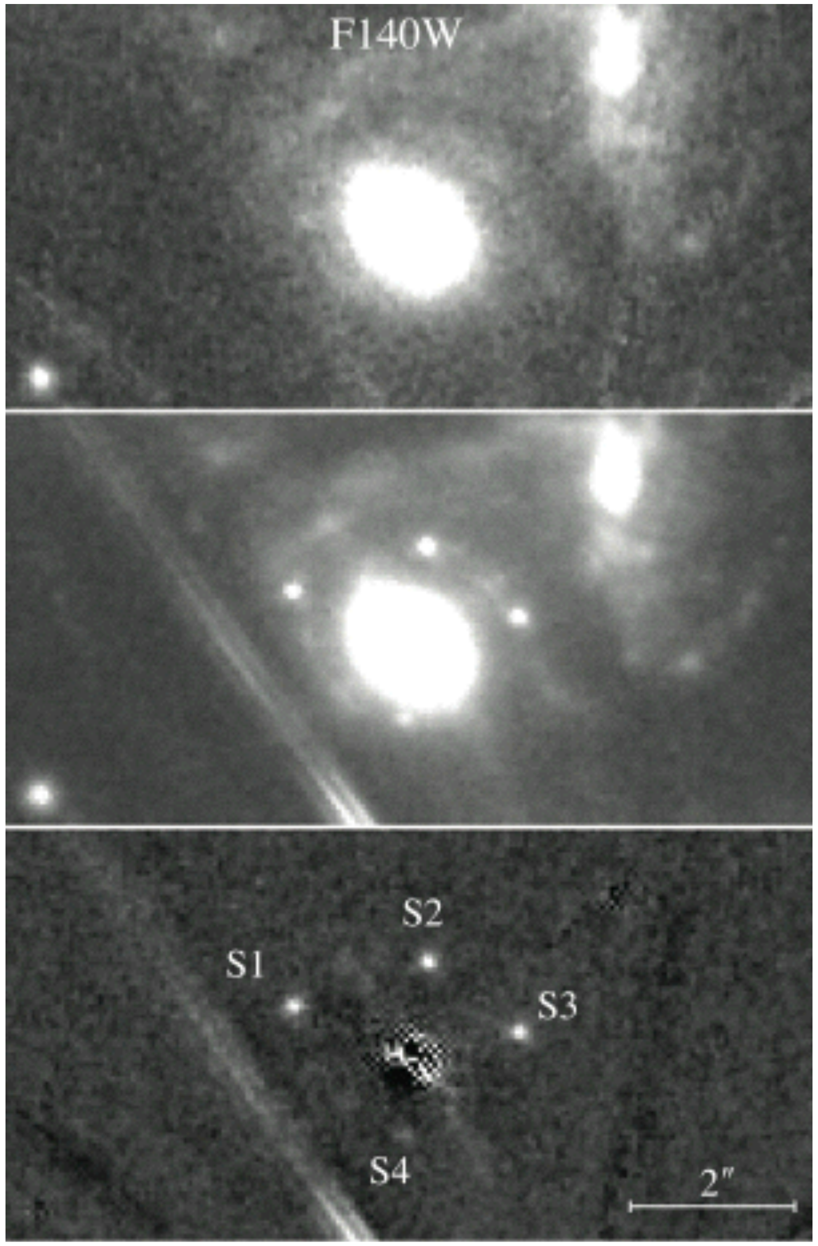
- Dec 2014: Imaging and spectroscopic follow-up of MACS1149 FoV
  - HFF, MOSFIRE, X-SHOOTER, DEIMOS, WFC3-G141

SN Refsdal

GLASS+ CLASH

GLASS+ HFF Dec ‘14

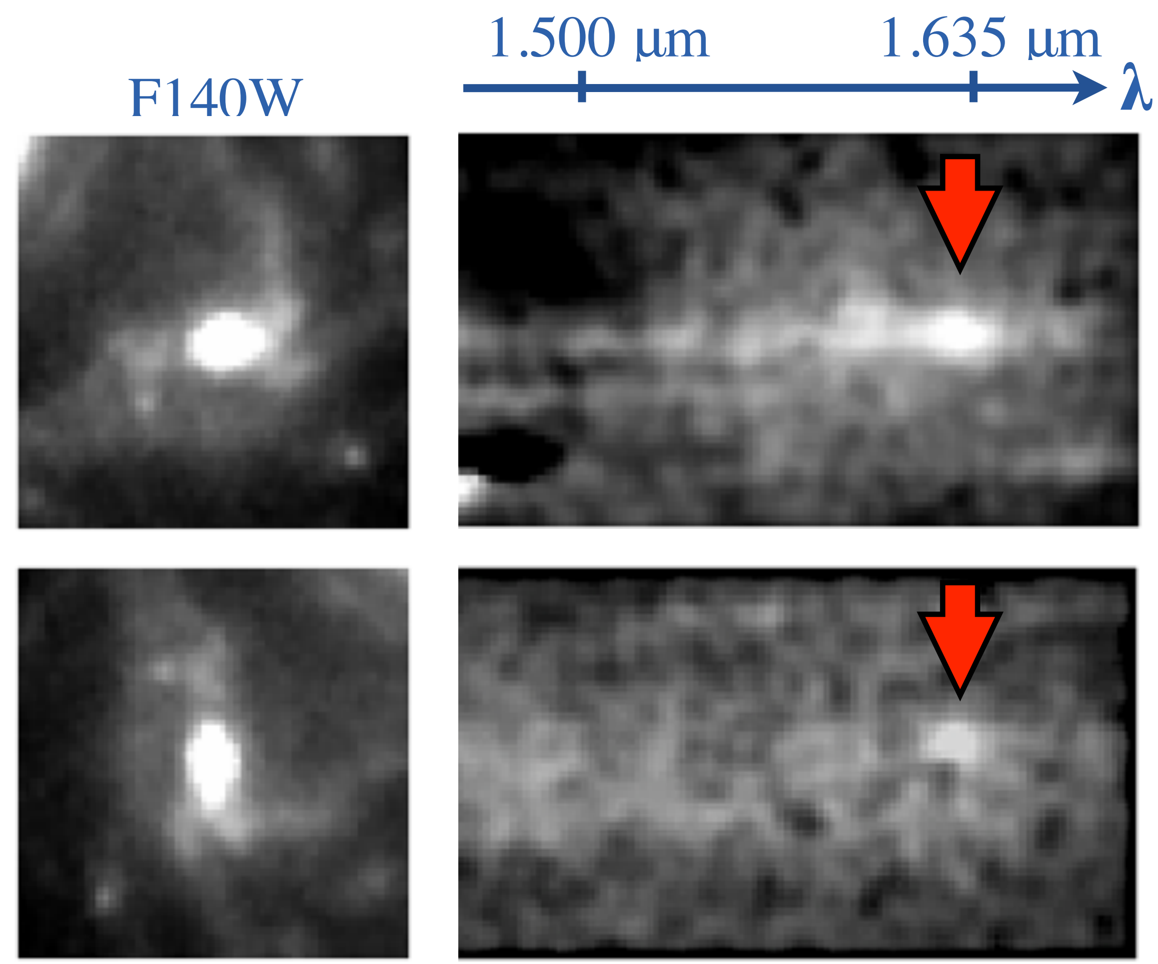
Diff.



Kelly+15

G141 PA1

G141 PA2

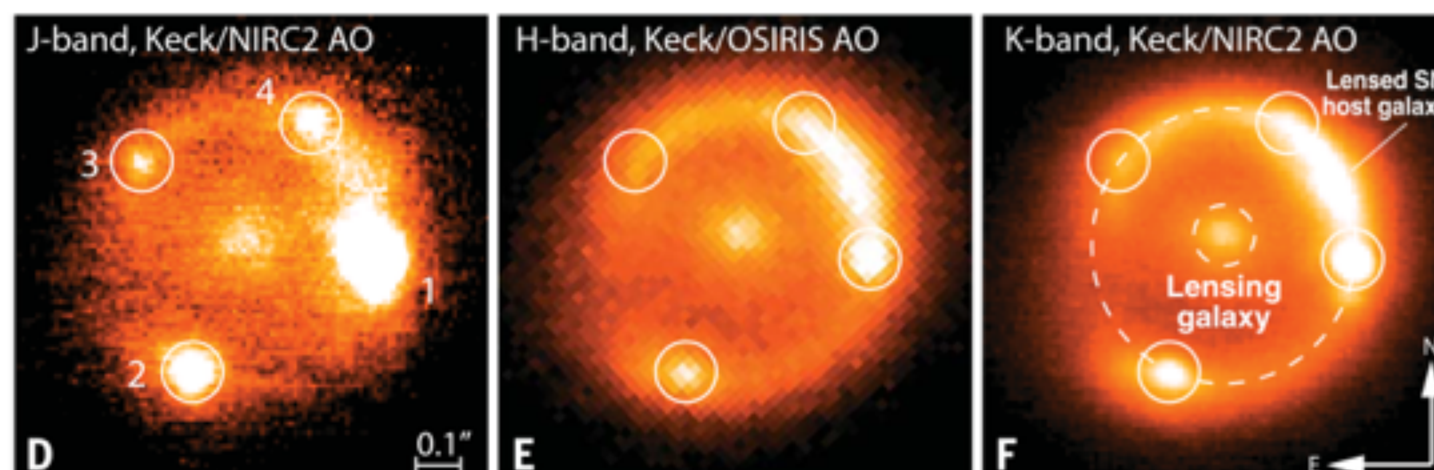


$z_{grism}(H\alpha) = 1.491$

Goobar et al. (2017)

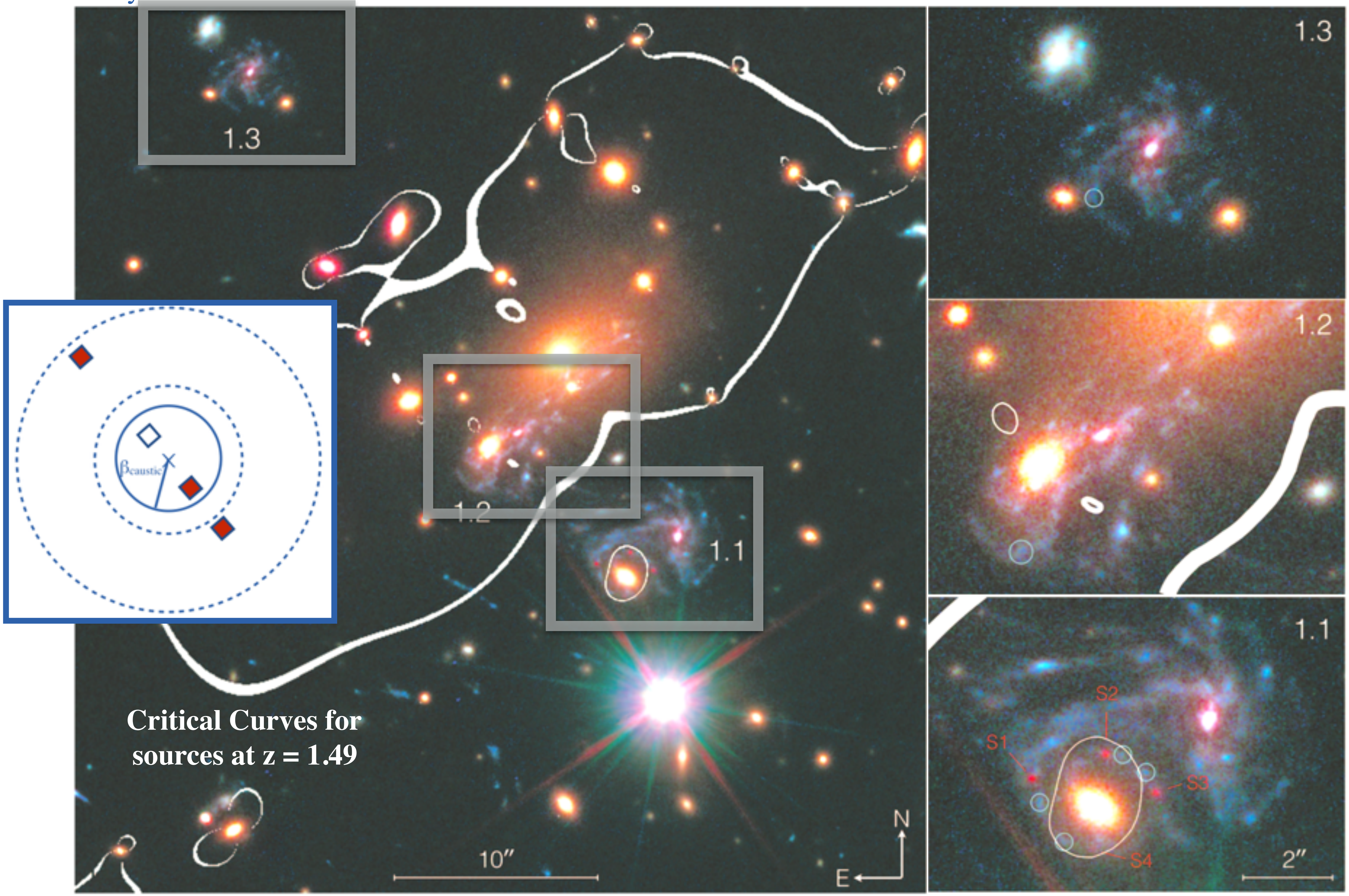
$z_{SNIa} = 0.409$

$z_{lens} = 0.216$



SN Refsdal - Multiple images

Kelly+2015



So in summary…

- The multiple images occurring from a point mass lens are given by

$$\theta_\pm = \frac{\beta}{2}\left[1 \pm \sqrt{1 + \frac{4\theta_E^2}{\beta^2}}\right]$$

- The lens equation for the more general 'spherically symmetric lens' is

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \langle\kappa(\theta)\rangle\boldsymbol{\theta} \quad \text{where} \quad \langle\kappa(\theta)\rangle = \frac{1}{\pi\theta^2}\int_{\theta'<\theta} d^2\theta' \kappa(\theta') = \frac{M(R = D_L\theta)}{\pi D_L^2\theta^2 \Sigma_{cr}}$$

- This leads to the lens equations for the CIS and SIS:

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \frac{\theta_0}{\theta^2}\left[\sqrt{\theta^2 + \theta_{core}^2} - \theta_{core}\right]\boldsymbol{\theta} \qquad \boldsymbol{\beta} = \boldsymbol{\theta}\left[1 - \frac{\theta_0}{|\theta|}\right]$$

- Solving for $\beta = 0$ reveals the multiple images for these lens models
- *Caustics* and *Critical curves* describe the source and image positions of multiple-image geometries in the source and lens (image) planes, respectively.
- SN Refsdal: a spectacular case of multiple images on galaxy and lens scales.