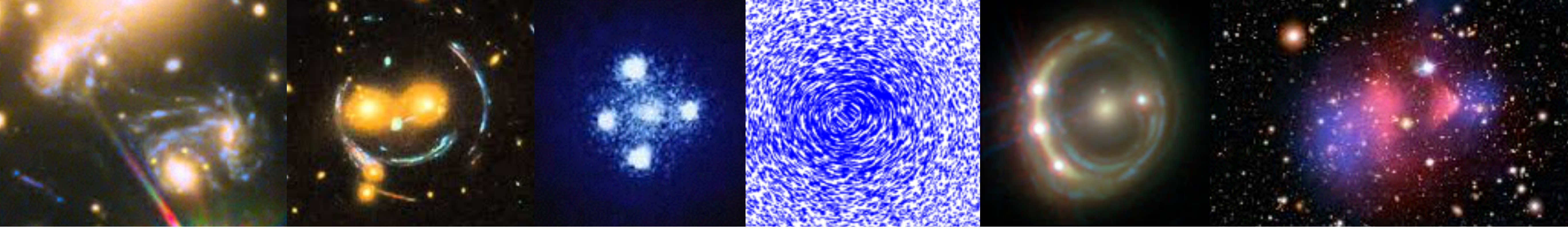

PHY-765 SS19 Gravitational Lensing Week 3

The Lens Equation

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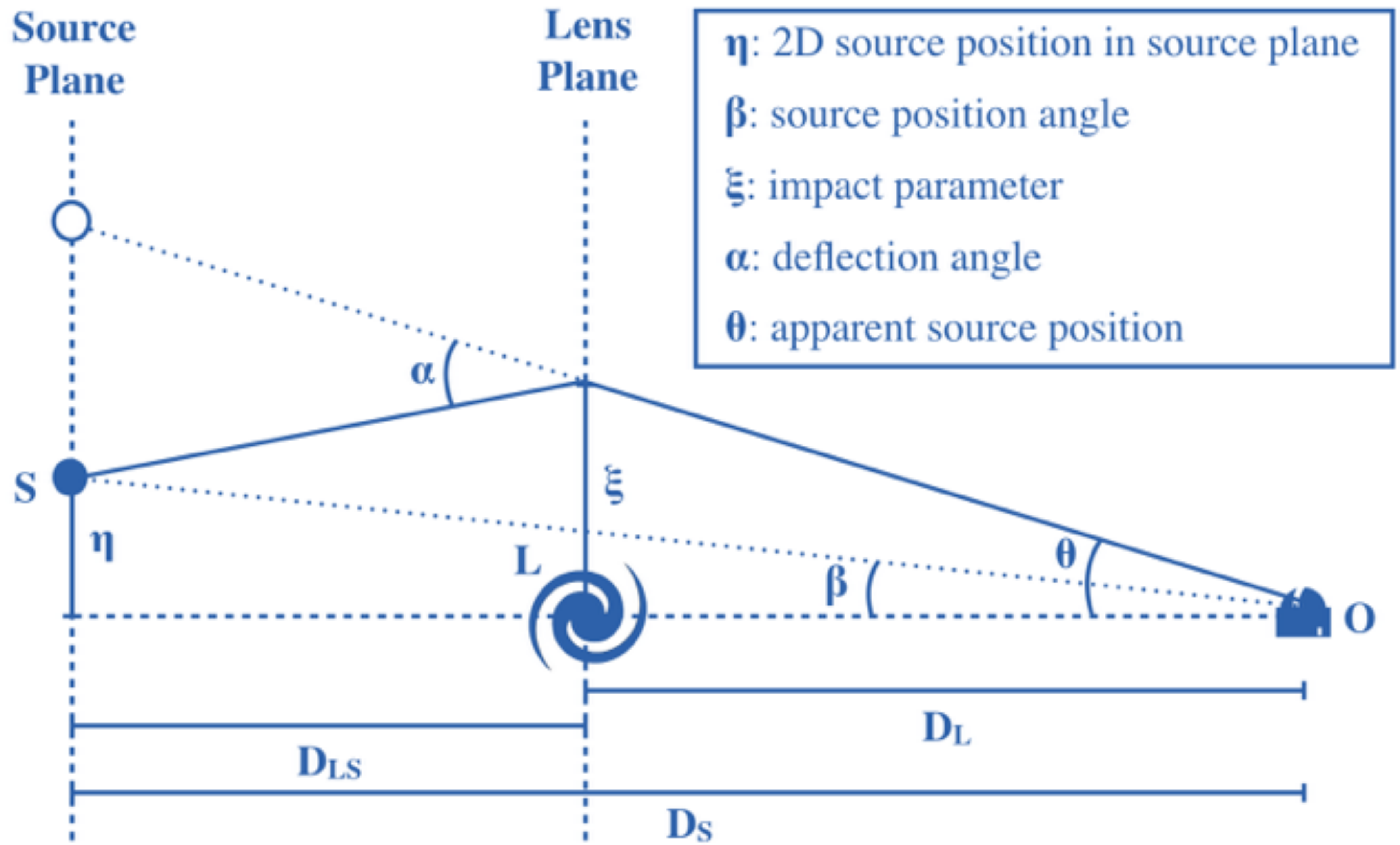
Last week - what did we learn?

- Overview of lens-geometries



- Derived the Newtonian and GR deflection angles for point masses

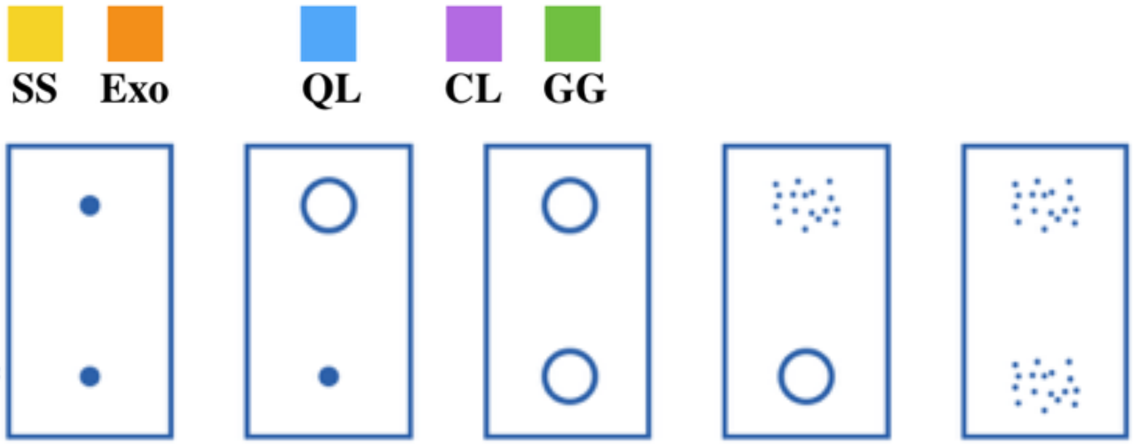
$$\alpha_N = \frac{2MG}{\xi c^2}$$

$$\alpha_{\rm GR} = 2 \times \alpha_N = \frac{4MG}{\xi c^2}$$

The aim of today

IB

- Use the geometry and the idea about deflection angles, α, to derive the general *lens equation*:
  - The main equation for describing GL
  - Relates source plane positions with lens plane positions
  - Accounts for relative distances of sources more explicitly
  - Describes observable phenomena like multiple images and time delays to be explored in the coming weeks



The point source deflection angle

- The deflection angle around a point source of mass *M* is given by

$$\hat{\alpha} = \frac{4MG}{\xi c^2} = \frac{2R_S}{\xi} \qquad \text{for} \quad \xi \gg R_S$$

- where $R_S \equiv \frac{2MG}{c^2}$ is the Schwarzschild radius
  - i.e., the radius of an object of mass *M* where light cannot escape the potential
- Hence, the deflection angle is small and we can infer that

$$\frac{\phi_N}{c^2} \ll 1 \qquad \text{where} \quad \phi_N = \frac{MG}{r}$$

- Perturbation theory linearizes Einstein's field equations as the Minkowski metric (*M*=0) plus a small 'perturbation' from the mass *M*.
- The Schwarzschild metric introduced last week:

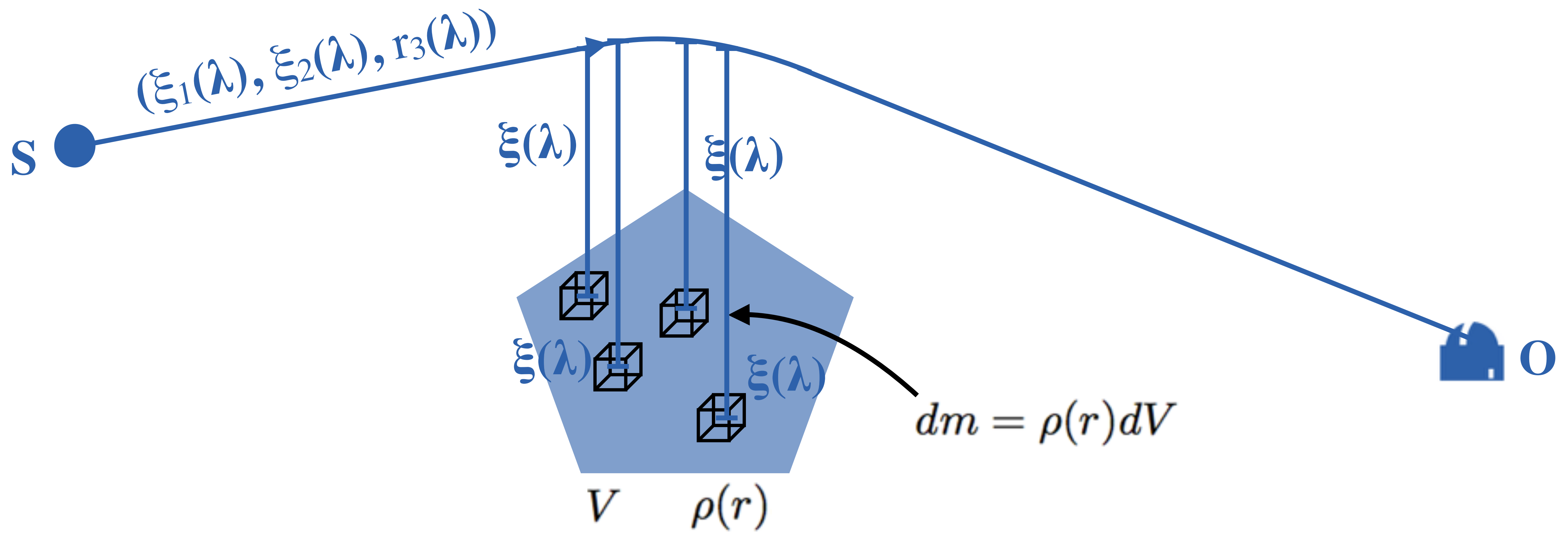
$$g_{00} = c^2\left(1 - \frac{2GM}{rc^2}\right)$$

$$g_{ij} = -\delta_{ij}\left(1 + \frac{2GM}{rc^2}\right)$$

- Thus, effects in GR space-time becomes 'linearly additive'

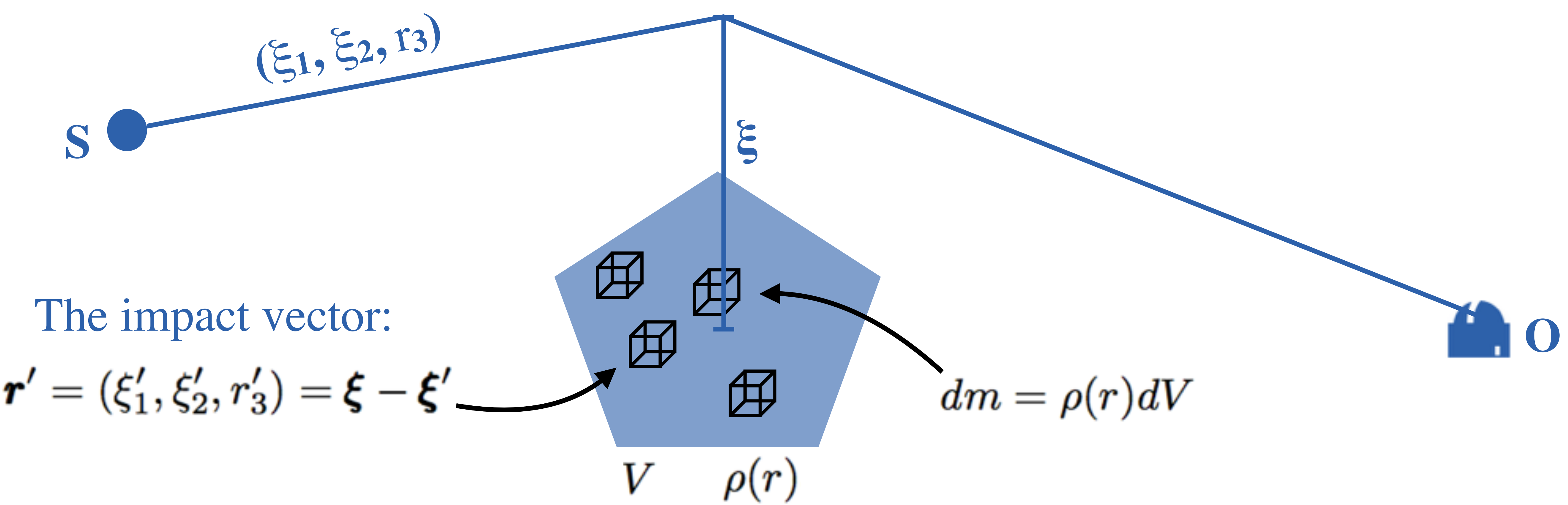
The deflection from a mass distribution

- Consider a mass distribution with volume V and volume density ρ(r)
- This volume can be divided into parcels of *dm*



The deflection from a mass distribution

- Consider a mass distribution with volume V and volume density ρ(r)
- This volume can be divided into parcels of *dm*
- Assuming a “thin lens”
  - Deflection happens in the 2D lens plane
  - Impact parameter does not depend on the affine parameter, λ
- Thin lens approximation true for essentially all astrophysical applications



The deflection from an mass distribution

- And since all the individual terms can be summed we then have:

$$\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \frac{4G}{c^2} \sum dm(\xi'_1, \xi'_2, r'_3) \frac{\boldsymbol{\xi} - \boldsymbol{\xi}'}{|\boldsymbol{\xi} - \boldsymbol{\xi}'|^2}$$

⬅ Direction

⬅ Amplitude

$$\hat{\alpha} = \frac{4MG}{\xi c^2}$$

The deflection from an mass distribution

- And since all the individual terms can be summed we then have:

$$\begin{aligned}\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) &= \frac{4G}{c^2}\sum dm(\xi'_1,\xi'_2,r'_3)\frac{\boldsymbol{\xi}-\boldsymbol{\xi}'}{|\boldsymbol{\xi}-\boldsymbol{\xi}'|^2} \\ &= \frac{4G}{c^2}\int d^2\xi' \int dr'_3\rho(\xi'_1,\xi'_2,r'_3)\frac{\boldsymbol{\xi}-\boldsymbol{\xi}'}{|\boldsymbol{\xi}-\boldsymbol{\xi}'|^2}\end{aligned}$$

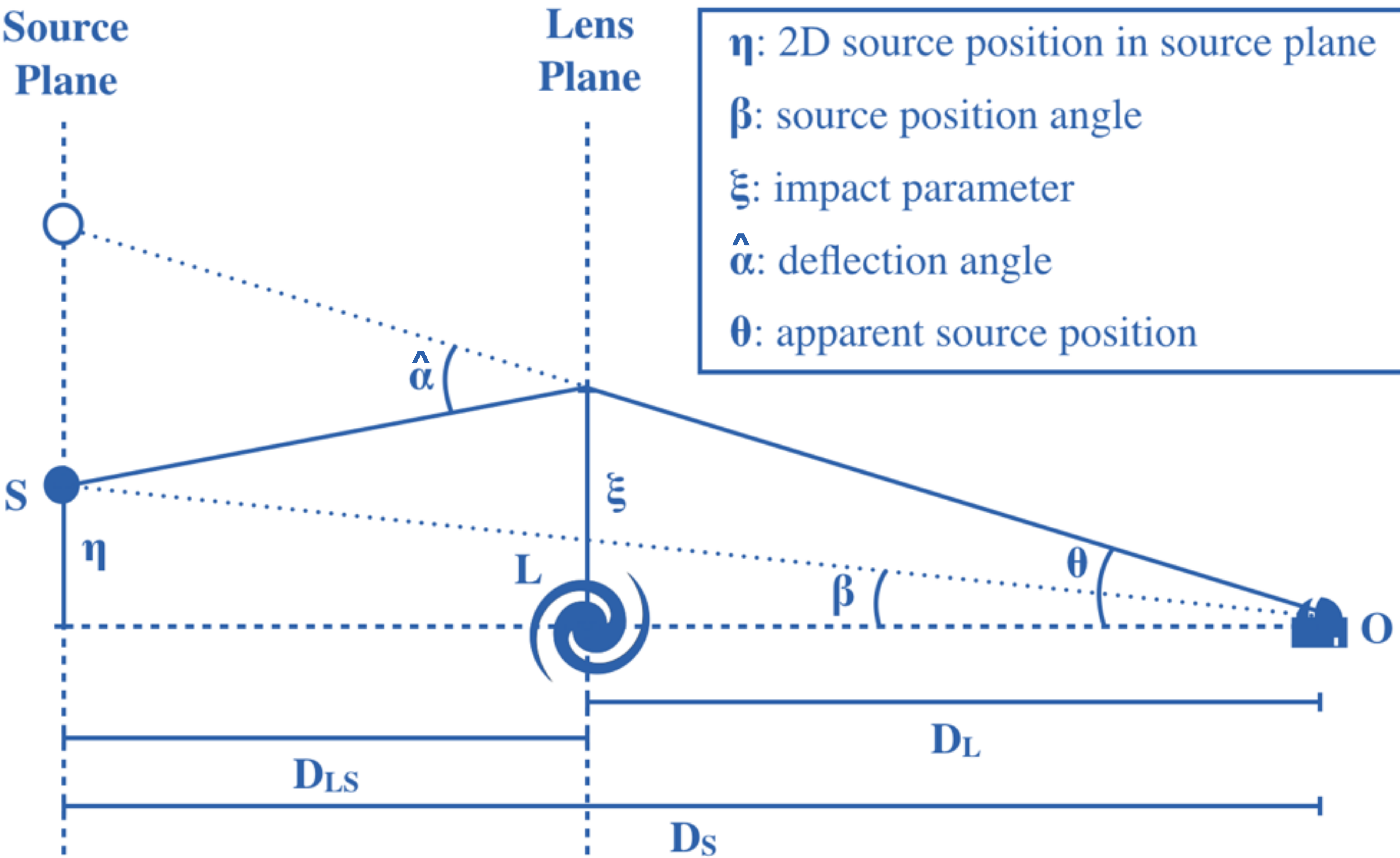
- As dV is spanned by $d\xi_1$, $d\xi_2$ and dr_3 and $\boldsymbol{\xi} = (\xi_1, \xi_2)$
- Defining the *surface mass density*:

$$\Sigma(\boldsymbol{\xi}) \equiv \int dr_3\, \rho(\xi_1, \xi_2, r_3)$$

- Gives us

$$\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \frac{4G}{c^2}\int d^2\xi'\, \Sigma(\boldsymbol{\xi}')\frac{\boldsymbol{\xi}-\boldsymbol{\xi}'}{|\boldsymbol{\xi}-\boldsymbol{\xi}'|^2}$$

The lens geometry



- **η**: 2D source position in source plane
- **β**: source position angle
- **ξ**: impact parameter
- **α̂**: deflection angle
- **θ**: apparent source position

The lens geometry

- We know the deflection is small such that $\sin\hat{\alpha} \sim \hat{\alpha} \sim \tan\hat{\alpha}$

$$\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \frac{\boldsymbol{a}}{D_{\mathrm{LS}}} \qquad \boldsymbol{\theta} = \frac{\boldsymbol{a}+\boldsymbol{\eta}}{D_{\mathrm{S}}} \qquad \boldsymbol{\eta} = \boldsymbol{\beta} D_{\mathrm{S}} \qquad \boldsymbol{\xi} = D_{\mathrm{L}}\boldsymbol{\theta}$$

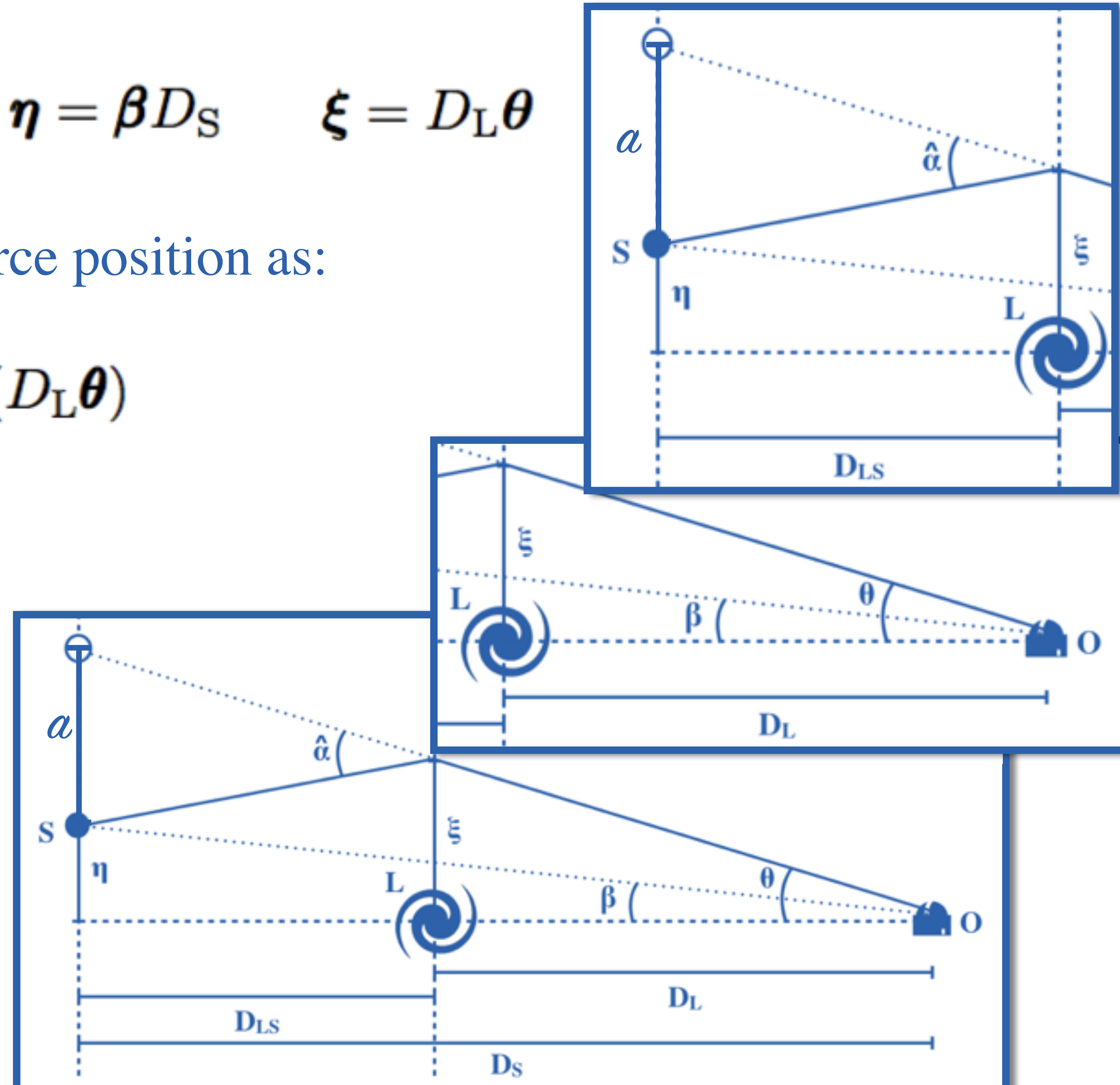
- We can then express the source position as:

(Exercise 3.1) $$\boldsymbol{\beta} = \boldsymbol{\theta} - \frac{D_{\mathrm{LS}}}{D_{\mathrm{S}}}\hat{\boldsymbol{\alpha}}(D_{\mathrm{L}}\boldsymbol{\theta})$$

- Defining the *scaled deflection angle* as:

$$\boldsymbol{\alpha}(\boldsymbol{\theta}) = \frac{D_{\mathrm{LS}}}{D_{\mathrm{S}}}\hat{\boldsymbol{\alpha}}(D_{\mathrm{L}}\boldsymbol{\theta})$$

- We have…



The lens equation

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \boldsymbol{\alpha}(\boldsymbol{\theta})$$

- The lens equation describes the non-linear mapping $\boldsymbol{\theta} \rightarrow \boldsymbol{\beta}$
- A source with true position $\boldsymbol{\beta}$ on the sky can be seen by an observer to be located at angular position $\boldsymbol{\theta}$ under the deflection $\boldsymbol{\alpha}(\boldsymbol{\theta})$.
- If, for fixed $\boldsymbol{\beta}$, there are multiple solutions satisfying the lens equation, the source will be observed at multiple locations on the sky, i.e., the lens produces multiple images of the source.

Critical Surface Mass Density & Convergence

- Defining
  - The *critical surface mass density*

$$\Sigma_{\rm cr} \equiv \frac{c^2}{4\pi G}\frac{D_{\rm S}}{D_{\rm L}D_{\rm LS}}$$

  - The *dimensionless surface mass density* or *convergence*

$$\kappa(\boldsymbol{\theta}) \equiv \frac{\Sigma(D_{\rm L}\boldsymbol{\theta})}{\Sigma_{\rm cr}}$$

- If a mass distribution has $\kappa > 1$ somewhere it can produce multiple images
  - Hence, $\Sigma_{\rm cr}$ is the characteristic value dividing 'weak' and 'strong' lensing
- We can then express the scaled deflection angle in terms of $\boldsymbol{\theta}$ instead of $\boldsymbol{\xi}$

$$\boldsymbol{\alpha}(\boldsymbol{\theta}) = \frac{1}{\pi}\int d^2\theta'\kappa(\boldsymbol{\theta}')\frac{\boldsymbol{\theta}-\boldsymbol{\theta}'}{|\boldsymbol{\theta}-\boldsymbol{\theta}'|^2}$$

(Exercise 3.2)

Relating deflection angle to deflection potential

- For any vector it can be shown that

$$\nabla \ln |\boldsymbol{\theta}| = \boldsymbol{\theta}/|\boldsymbol{\theta}|^2$$

(Exercise 3.3)

- So we can write the deflection potential as

$$\psi(\boldsymbol{\theta}) = \frac{1}{\pi} \int d^2\theta' \kappa(\boldsymbol{\theta}') \ln |\boldsymbol{\theta} - \boldsymbol{\theta}'|$$

- given that

$$\boldsymbol{\alpha} = \nabla \psi$$

- Using $\nabla^2 \ln |\boldsymbol{\theta}| = 2\pi \delta_{\mathrm{D}}(\boldsymbol{\theta})$ implies $\nabla^2 \psi = 2\kappa$

- Hence, the mapping of $\boldsymbol{\theta} \rightarrow \boldsymbol{\beta}$ is a gradient mapping

- Similarities to standard 3D gravity when it's realized that

$$\psi, \boldsymbol{\alpha}, \kappa \quad \text{corresponds to} \quad \phi_{\mathrm{N}}, \frac{d^2\boldsymbol{r}}{dt^2}, \rho$$

Last week:

$$\frac{d^2 x^i}{dt^2} = -\frac{2MGx^i}{r^3}$$

$$\phi_N = \frac{MG}{r}$$

Back to the point source

- The point source has a surface density set by

$$\Sigma = M\delta_{\mathrm{D}}^2(\boldsymbol{\theta})/D_{\mathrm{L}}^2$$

- The δ_D sets $\boldsymbol{\theta}' = 0$ in the surface integrals, such that

$$\boldsymbol{\psi}(\boldsymbol{\theta}) = \frac{1}{\pi}\int d^2\theta' \kappa(\boldsymbol{\theta}') \ln|\boldsymbol{\theta} - \boldsymbol{\theta}'| \quad \rightarrow \quad \boldsymbol{\psi}(\boldsymbol{\theta}) = \frac{4MGD_{\mathrm{LS}}}{c^2 D_{\mathrm{S}} D_{\mathrm{L}}} \ln|\boldsymbol{\theta}|$$

- And the deflection angle becomes

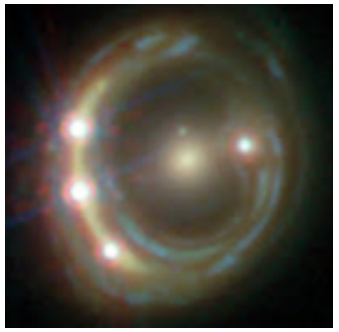
$$\boldsymbol{\alpha}(\boldsymbol{\theta}) = \frac{1}{\pi}\int d^2\theta' \kappa(\boldsymbol{\theta}') \frac{\boldsymbol{\theta} - \boldsymbol{\theta}'}{|\boldsymbol{\theta} - \boldsymbol{\theta}'|^2} \quad \rightarrow \quad \boldsymbol{\alpha}(\boldsymbol{\theta}) = \frac{4MGD_{\mathrm{LS}}}{c^2 D_{\mathrm{S}} D_{\mathrm{L}}} \frac{\boldsymbol{\theta}}{|\boldsymbol{\theta}|^2}$$

- Inserting this into the lens equation we get

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \frac{4MGD_{\mathrm{LS}}}{c^2 D_{\mathrm{S}} D_{\mathrm{L}}} \frac{\boldsymbol{\theta}}{|\boldsymbol{\theta}|^2}$$

- If the source is directly behind the lens ($\boldsymbol{\beta} = 0$) the Einstein radius appears:


$$|\boldsymbol{\theta}| = \theta_E \equiv \sqrt{\frac{4MG}{c^2}\frac{D_{\mathrm{LS}}}{D_{\mathrm{S}} D_{\mathrm{L}}}}$$

Distances in cosmology

- So far it has been glossed over what is meant with ‘distances’
- Cosmology deals with multiple distances, all related to each other via z
  - Co-moving distance, D^{cm}
  - Luminosity distance, D^{lum}
  - Angular diameter distance, D^{ang}
- So what distances should we use?
- All of the above equations hold under the assumption that $\mathrm{D} \equiv \mathrm{D}^{\mathrm{ang}}$
  - For Euclidean geometry $D_{\mathrm{LS}} = D_{\mathrm{S}} - D_{\mathrm{L}}$; Not for angular diameter distances
- At high redshift co-moving distances are needed:

$$D^{\mathrm{cm}} = c \int_0^z \frac{dz'}{H(z')}$$

$$D^{\mathrm{ang}}_{\mathrm{S}} \rightarrow \frac{D^{\mathrm{cm}}(z_{\mathrm{S}})}{1+z_{\mathrm{S}}} \qquad D^{\mathrm{ang}}_{\mathrm{L}} \rightarrow \frac{D^{\mathrm{cm}}(z_{\mathrm{L}})}{1+z_{\mathrm{L}}} \qquad D^{\mathrm{ang}}_{\mathrm{LS}} \rightarrow \frac{D^{\mathrm{cm}}(z_{\mathrm{S}}) - D^{\mathrm{cm}}(z_{\mathrm{L}})}{1+z_{\mathrm{S}}}$$

So in summary…

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \boldsymbol{\alpha}(\boldsymbol{\theta})$$

$$\boldsymbol{\alpha}(\boldsymbol{\theta}) = \frac{1}{\pi} \int d^2\theta' \kappa(\boldsymbol{\theta}') \frac{\boldsymbol{\theta} - \boldsymbol{\theta}'}{|\boldsymbol{\theta} - \boldsymbol{\theta}'|^2}$$

- Along the way we defined:

$$\Sigma_{\rm cr} \equiv \frac{c^2}{4\pi G} \frac{D_{\rm S}}{D_{\rm L} D_{\rm LS}} \qquad \kappa(\boldsymbol{\theta}) \equiv \frac{\Sigma(D_{\rm L}\boldsymbol{\theta})}{\Sigma_{\rm cr}} \qquad \theta_E \equiv \sqrt{\frac{4MG}{c^2} \frac{D_{\rm LS}}{D_{\rm S} D_{\rm L}}}$$