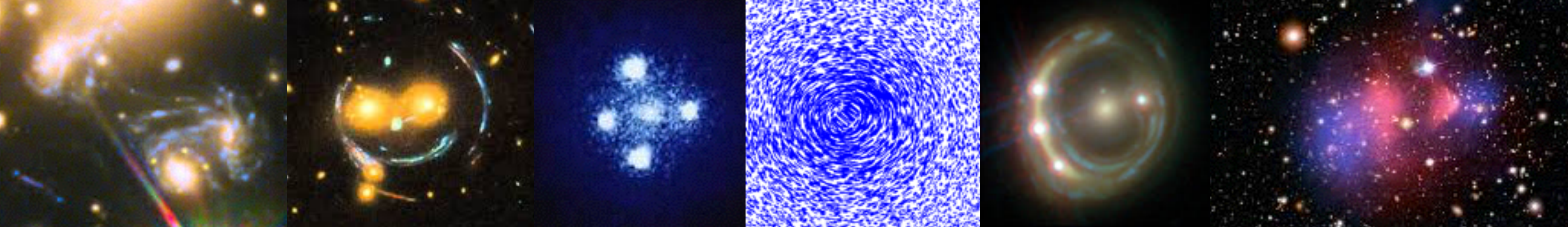

PHY-765 SS19 Gravitational Lensing Week 2

GL Geometry & Light Deflection

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Last week - what did we learn?

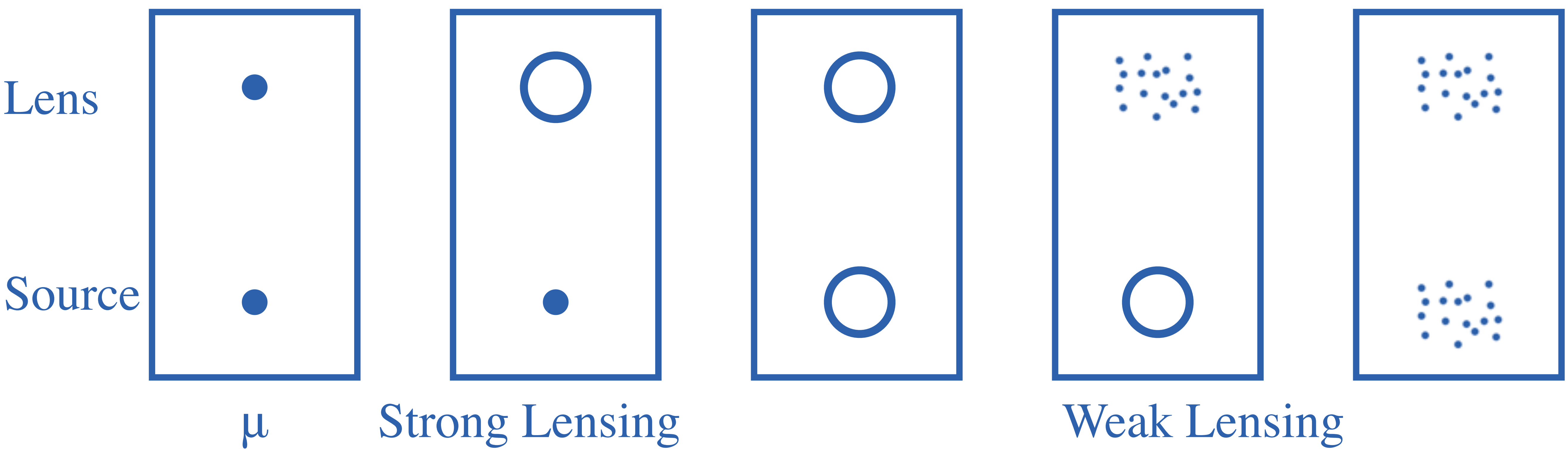
- History of GL including early predictions including:
  - Light is affected by gravity
  - Deflections of 10s of arc sec for galaxy lenses
  - Useful for lens mass estimates
  - Spectroscopy is a key for identifying lenses
- People were considering deflection of light in Newtonian gravity (<1915):
  - $\alpha_N = 2GM / c^2r$
- GR came along and changed this deflection to (>1915)
  - $\alpha_{GR} = 2 \times \alpha_N = 4GM / c^2r$
- Growing importance of GL over the last 40 years (still growing)

The aim of today

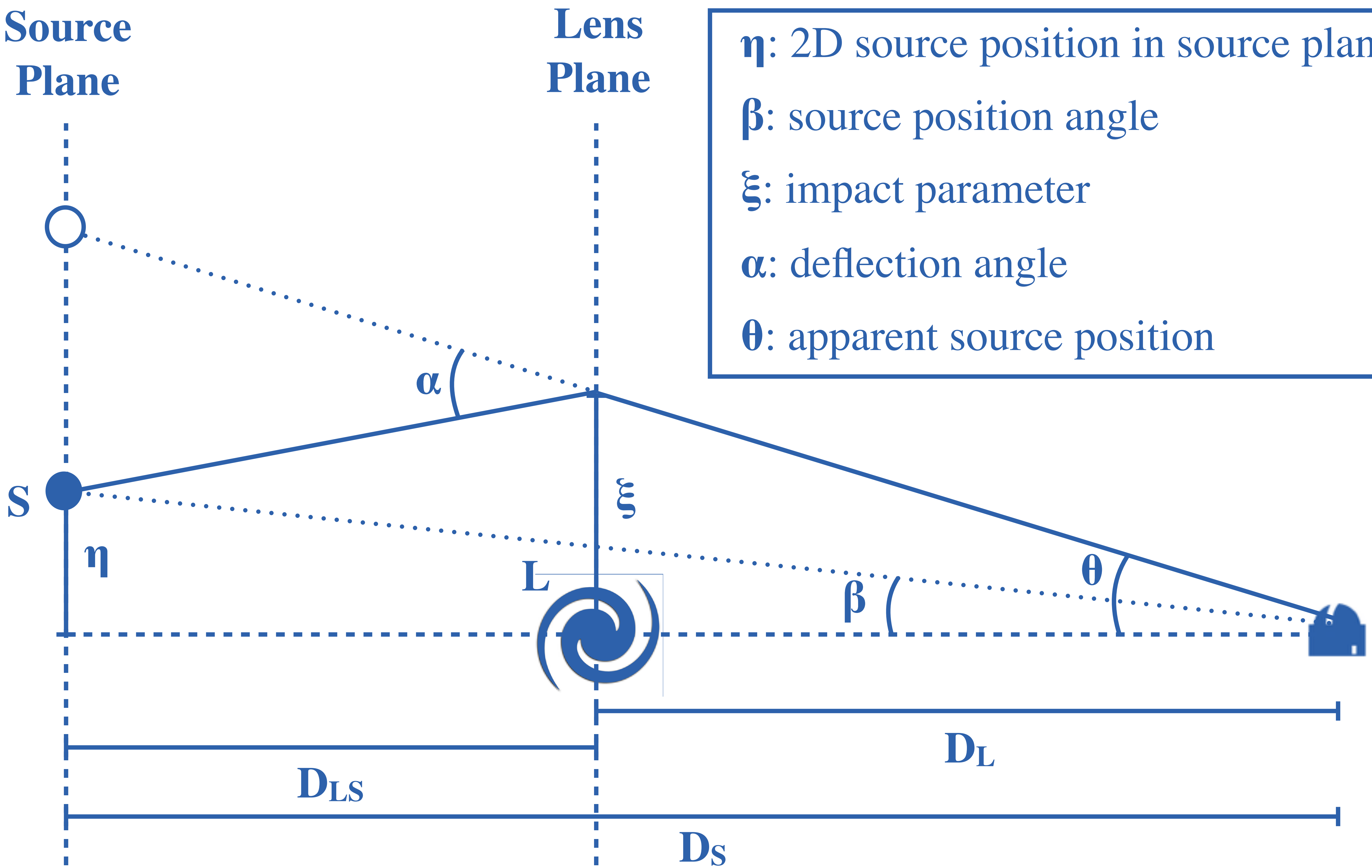
- The Geometry of Gravitational Lensing:
  - Schematic of angles, distances and light-paths in GL
- Light deflection:
  - How did they arrive at the Newtonian deflection angle?
  - What changed things for the GR deflection angle?

Gravitational Lensing Course “Roadmap”

Introduction & Basic theory	**IB**
Cluster lensing	**CL**
QSO lensing	**QL**
Galaxy-Galaxy lensing	**GG**
Star-Star microlensing	**SS**
Exoplanet searches with microlensing	**Exo**
Wide-field weak lensing	**WF**
Power Spectrum lensing analysis	**PS**



Gravitational Lensing Geometry



η: 2D source position in source plane

β: source position angle

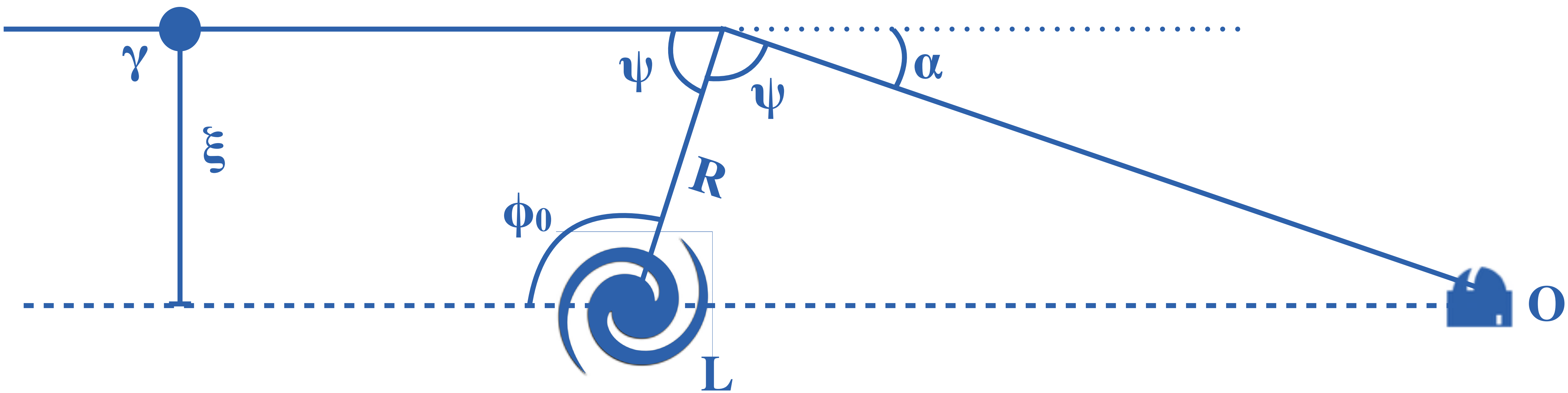
ξ: impact parameter

α: deflection angle

θ: apparent source position

The Newtonian Deflection Angle, α_N

- Last week we heard that, e.g., Newton, Cavendish, Laplace and Soldner all considered deflection of light in Newtonian gravity. But how?



- Consider a point particle, γ of mass m, deflected by a lens, L, with mass M
- Position a polar coordinate system (ϕ,R) with origin on the lens
- Let ϕ_0 be the angle at closest approach
- We know from Newtonian Gravity that in 3D the particle γ obeys

$$m\frac{d^2\vec{r}}{dt^2} = -\frac{mMG}{r^2}\vec{r}$$

The Newtonian Deflection Angle, α_N

- But the motion considered here is confined to 2D (ϕ,R), so decomposing

$$m\frac{d^2\vec{r}}{dt^2} = -\frac{mMG}{r^2}\vec{r}$$

- into the polar coordinate system (ϕ,R) we have

$$\hat{R}\left[\ddot{R} - R\dot{\phi}^2\right] + \hat{\phi}\frac{1}{R}\frac{d}{dt}\left[R^2\dot{\phi}\right] = -\frac{MG}{R^2}\hat{R} \qquad \text{(Exercise 2.1)}$$

- Note: *m* drops out, i.e. this expression is independent of particle mass
  - Convenient given that we want to consider the mass-less photon, γ
- (specific) Angular momentum is a conserved quantity

$$R^2\dot{\phi} = \frac{|L|}{m} \equiv J_z$$

- Hence

$$\frac{d}{dt}\left[R^2\ddot{\phi}\right] = \frac{d}{dt}\left[J_z\right] = 0$$

The Newtonian Deflection Angle, α_N

- This gives

$$\hat{R}\left[\ddot{R} - R\dot{\phi}^2\right] = -\frac{MG}{R^2}\hat{R}$$

$$\ddot{R} - \frac{1}{R^3}\left(R^4\dot{\phi}^2\right) = -\frac{MG}{R^2}$$

$$\ddot{R} - \frac{J_z^2}{R^3} = -\frac{MG}{R^2}$$

- Using that

$$\dot{R} = \frac{J_z R'}{R^2}$$

$$\ddot{R} = \frac{J_z^2}{R^2}\left[\frac{R''}{R^2} - 2\frac{R'^2}{R^3}\right]$$

(Exercise 2.2)

- The equation of motion becomes

$$\frac{R''}{R^2} - 2\frac{R'^2}{R^3} - \frac{1}{R} = -\frac{MG}{J_z^2}$$

The Newtonian Deflection Angle, α_N

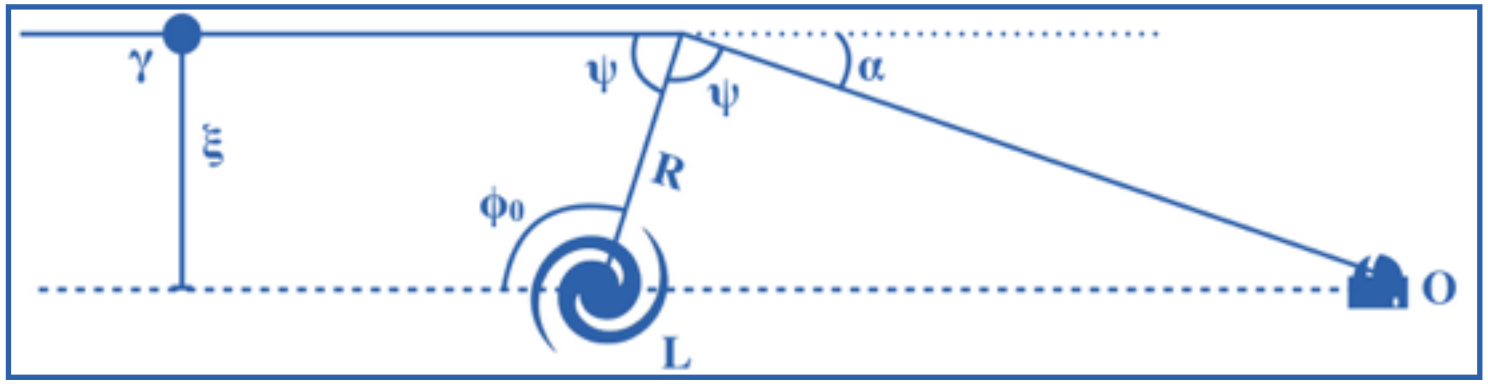
- Changing variable from *R* to *u*, where $u \equiv 1/R$ we get

$$u'' + u = \frac{MG}{J_z^2} \qquad \text{(Exercise 2.3)}$$

- This is an inhomogeneous second order differential equation
- The solution to this equation can be expressed by the cyclic function

$$\frac{1}{R(\phi)} = A \times \cos\left[\phi - \phi_0\right] + \frac{MG}{J_z^2}$$

- At very early times
  - R is large
  - ϕ is small



$$\tan\phi = \frac{\xi}{R} \simeq \phi \qquad \Rightarrow \qquad \frac{d}{dt}\phi = \dot{\phi} = -\frac{\xi}{R^2}\dot{R}$$

- Which means that at early times: $J_z = -\xi \dot{R}$

The Newtonian Deflection Angle, α_N

- But as we are looking at the photon the velocity is -*c* giving us that

$$J_z = \xi c$$

(-*c* as velocity is opposite *R*-direction)

- which can be inserted into the solution to the equation of motion

$$\frac{1}{R(\phi)} = A \times \cos[\phi - \phi_0] + \frac{MG}{\xi^2 c^2}$$

- Still need to determine *A* and ϕ_0. We will use "initial conditions" for this
- In the limit when R → ∞ then $\phi = 0$

$$\cos\phi_0 = -\frac{MG}{A\xi^2 c^2}$$

The Newtonian Deflection Angle, α_N

- Secondly we look at the initial velocity, i.e. differentiating wrt. *t*

$$\frac{d}{dt}\frac{1}{R} = \frac{d}{dt}A\cos(\phi-\phi_0)+\frac{d}{dt}\frac{MG}{\xi^2c^2}$$

$$\Downarrow$$

$$-\frac{\dot{R}}{R^2} = A\left(-\sin(\phi-\phi_0)\dot{\phi}\right)$$

$$= A\left(-\sin(\phi-\phi_0)\left(-\frac{\xi}{R^2}\dot{R}\right)\right)$$

$$\Downarrow$$

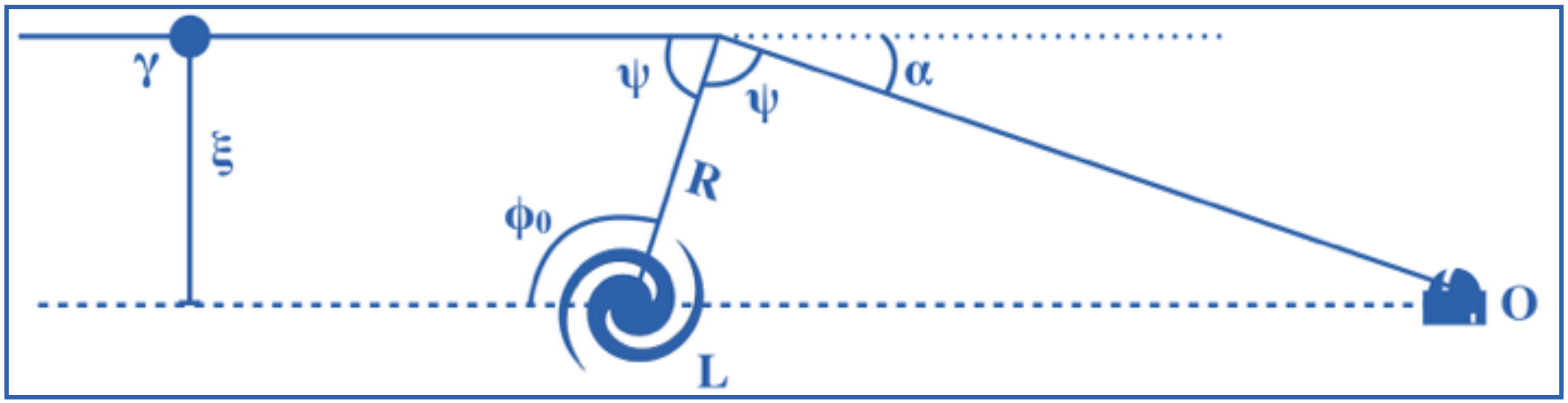
$$A = \frac{1}{\sin(\phi-\phi_0)\xi} \rightarrow \frac{1}{\sin(\phi_0)\xi}$$

- This gives us two equations to determine the two unknowns *A* and ϕ_0

The Newtonian Deflection Angle, α_N

- So we have the following:

$$\frac{1}{R(\phi)} = A \times \cos[\phi - \phi_0] + \frac{MG}{\xi^2 c^2} \qquad \cos\phi_0 = -\frac{MG}{A\xi^2 c^2} \qquad A = \frac{1}{\sin(\phi_0)\xi}$$



- What is the size of ϕ_0 without deflection? $\phi_0 = \pi/2$

$$A \simeq \frac{1}{\xi} \qquad \Rightarrow \qquad \cos\phi_0 = -\frac{MG}{\xi c^2}$$

- With deflection what is the size of ϕ_0 compared to $\phi = \pi/2$? $\phi \sim \pi/2 + \varepsilon$

- Taylor expanding this expression leads to:

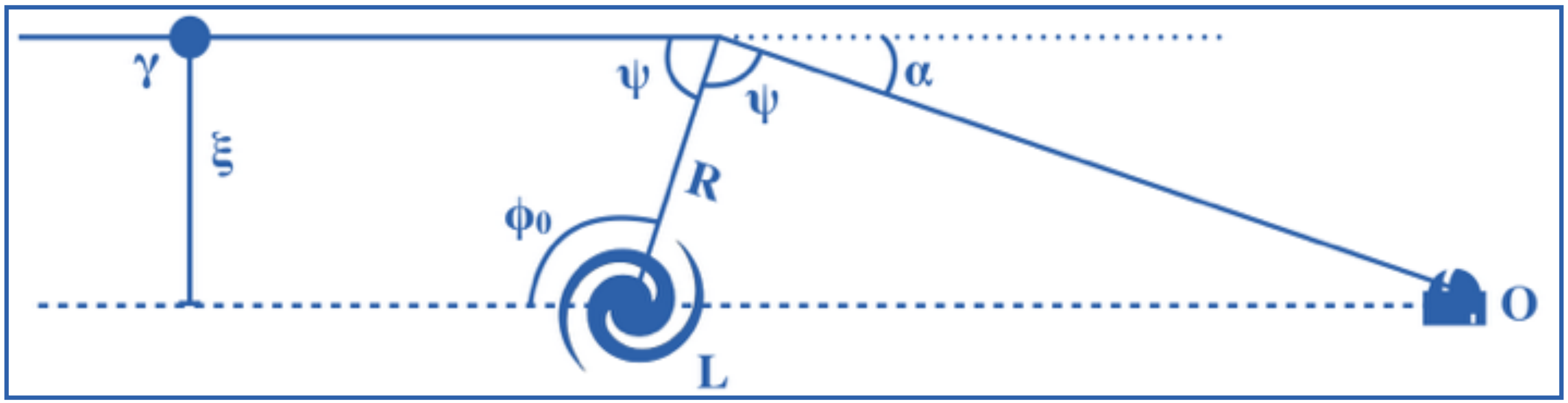
$$\begin{aligned}\cos\phi_0 &\simeq \cos(\pi/2) + (-\sin(\pi/2))(\phi_0 - \pi/2) + \frac{-\cos(\pi/2)}{2}(\phi_0 - \pi/2) + \ldots \\ &\simeq -(\phi_0 - \pi/2) = -\epsilon\end{aligned}$$

The Newtonian Deflection Angle, α_N

- Which then results in

$$\phi_0 = \frac{\pi}{2} + \frac{MG}{\xi c^2}$$

- From geometry we can express α



$$2\psi + \alpha = \pi \qquad\qquad (\pi - \phi_0) + \psi + \alpha = \pi$$

- So we have that

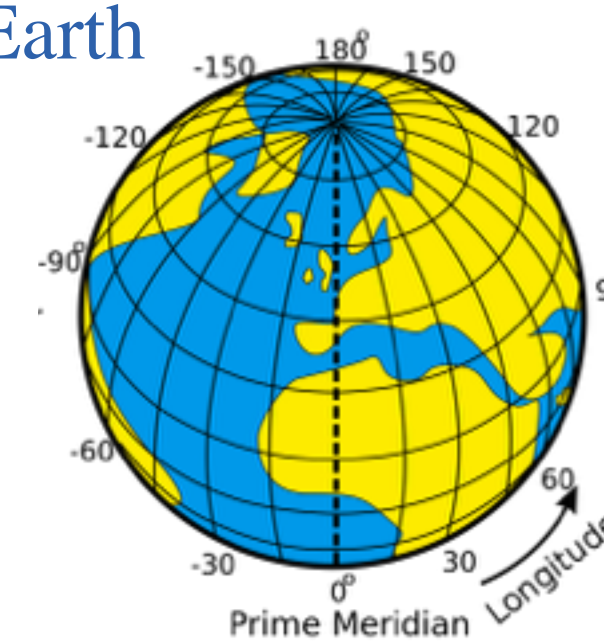
$$\alpha = 2\phi_0 - \pi$$

- Which combined with the above gives:

$$\boxed{\alpha_N = \frac{2MG}{\xi c^2}}$$

The GR Deflection Angle, α_{GR}

- Moving to GR, we want to describe the distortion by gravity in the curved space-time. Curvature analog to longitude and latitude on Earth



- Need to define some GR jargon:
  - $g_{\mu\nu}$: The metric tensor where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$
  - $\Gamma^{\beta}_{\mu\nu}$: The *affine connection*, i.e. Christoffel symbols

$$\Gamma^{\beta}_{\mu\nu} = \frac{g^{\beta\beta}}{2}\left[\frac{\partial g_{\beta\mu}}{\partial x^{\nu}} + \frac{\partial g_{\beta\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}}\right]$$

- We consider the geodesic equation (geodesic = “straight line”)

$$\frac{d^2x^i}{d\lambda^2} = -\Gamma^{i}_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}$$

- where $x^{\alpha} = (t, x, y, z)$ and β,μ,ν run over (0,1,2,3) and i,j,k over (1,2,3)
- In GR time t is not ‘special’ so we differentiate wrt. the *affine parameter* λ
- For instance the four-momentum is defined as $P = \frac{dx^{\alpha}}{d\lambda} \equiv p^{\alpha} = (E/c, \bar{p})$

The GR Deflection Angle, α_{GR}

- Using that

$$\frac{dx^i}{d\lambda} = \frac{dx^i}{dt}\frac{dt}{d\lambda} = \frac{dx^i}{dt}\frac{E}{c}$$

$$\frac{d^2x^i}{d\lambda^2} = \frac{E}{c}\frac{d}{dt}\left[\frac{E}{c}\frac{dx^i}{dt}\right] \simeq \frac{E^2}{c^2}\frac{d^2x^i}{dt^2}$$

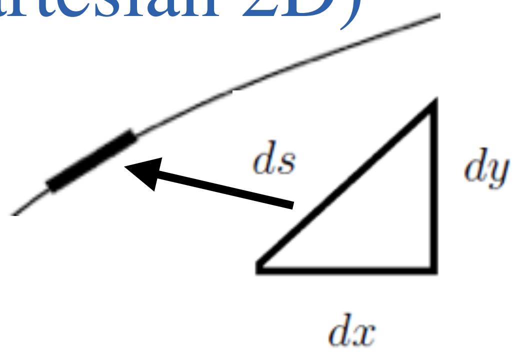
- We can write the geodesic equation as

$$\frac{E^2}{c^2}\frac{d^2x^i}{dt^2} = -\Gamma^i_{\mu\nu}p^\mu p^\nu$$

- This is the expression for a particle's motion in a given space-time
- Need to define the space-time through the 'metric'.
- The metric when deflection is induced by a point mass M is

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 00 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{pmatrix} \quad \text{where} \quad \begin{aligned} g_{00} &= c^2\left(1 - \frac{2GM}{rc^2}\right) \\ g_{ij} &= -\delta_{ij}\left(1 + \frac{2GM}{rc^2}\right) \end{aligned}$$

The GR Deflection Angle, α_{GR}



- The line element for this metric is (analog to 'Pythagoras' in cartesian 2D)

$$ds^2 = g_{00}\, dt^2 + g_{ij}\, dx^i\, dx^j = 0$$

- GR time dilation:

$$g_{00} = c^2 \left(1 - \frac{2GM}{rc^2}\right)$$

- Two events at the same location ($dx = 0$)

$$ds = \sqrt{c^2 \left(1 - \frac{2GM}{rc^2}\right)} dt \qquad \Rightarrow \qquad dt \simeq \frac{ds}{c}\left(1 + \frac{GM}{rc^2}\right)$$

- GR Length Contraction:

$$g_{ij} = -\delta_{ij}\left(1 + \frac{2GM}{rc^2}\right)$$

- Two events at the same time ($dt = 0$)

$$ds = \sqrt{-\left(1 + \frac{2GM}{rc^2}\right)} dx^i \qquad \Rightarrow \qquad dx^i \simeq ds\left(1 - \frac{GM}{rc^2}\right)$$

The GR Deflection Angle, α_{GR}

- Using the metric $g_{\mu\nu}$ we can derive the Christoffel symbols using

$$\Gamma^{\beta}_{\mu\nu} = \frac{g^{\beta\beta}}{2}\left[\frac{\partial g_{\beta\mu}}{\partial x^{\nu}} + \frac{\partial g_{\beta\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}}\right]$$

where

$$g_{00} = c^2\left(1 - \frac{2GM}{rc^2}\right)$$
$$g_{ij} = -\delta_{ij}\left(1 + \frac{2GM}{rc^2}\right)$$

- This enables us to express the geodesic equation in terms of the metric:

$$\frac{E^2}{c^2}\frac{d^2x^i}{dt^2} = -\Gamma^i_{\mu\nu}p^{\mu}p^{\nu}$$
$$= -\Gamma^i_{00}p^0p^0 - 2\Gamma^i_{0j}p^0p^j - \Gamma^i_{jk}p^jp^k$$
$$= -\frac{MGx^i}{r^3}\frac{E^2}{c^2} - 0 - \Gamma^i_{33}p^3p^3$$
$$= -\frac{MGx^i}{r^3}\frac{E^2}{c^2} - \frac{MGx^i}{r^3}\frac{E^2}{c^2}$$
$$= -\frac{2MGx^i}{r^3}\frac{E^2}{c^2}$$

Central term 2nd order
Last term only solution in z

using $p^z = E/c$ for photon

The GR Deflection Angle, α_{GR}

General Relativity

$$\frac{d^2x^i}{dt^2} = -\frac{2MGx^i}{r^3}$$

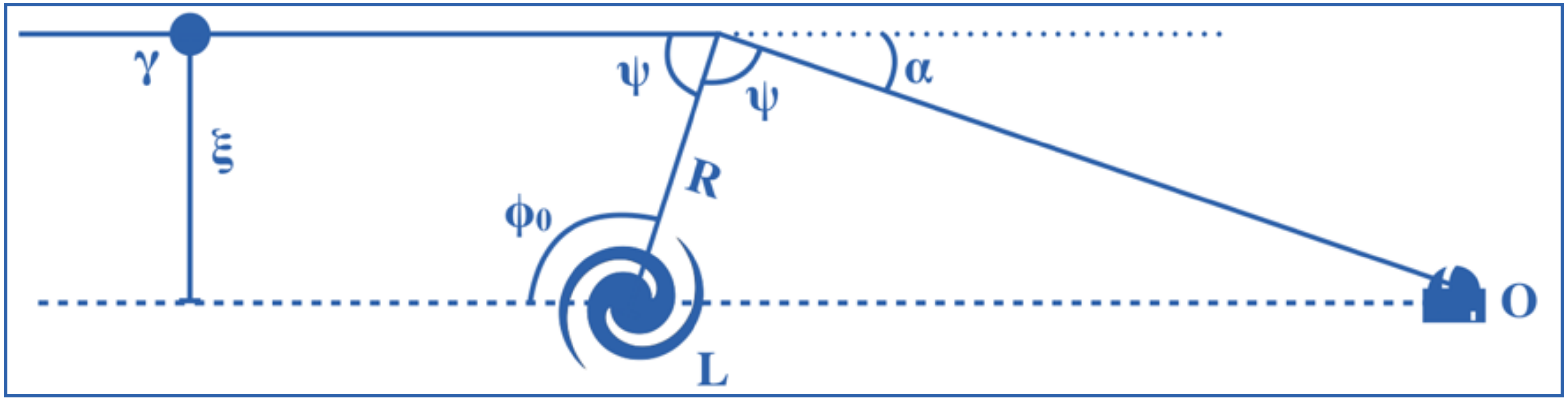
Newtonian Gravity

$$\frac{d^2\vec{r}}{dt^2} = -\frac{MG}{r^2}\vec{r}$$

- Hence, by realizing that x^i just represent the spatial vector of the photon in GR, we can move on from here, following the Newtonian derivation of α_N step by step carrying through the factor of 2 and eventual arrive at:

$$\alpha_{\rm GR} = 2 \times \alpha_N = \frac{4MG}{\xi c^2}$$

So in summary…



$$\alpha_N = \frac{2MG}{\xi c^2}$$

$$\alpha_{\rm GR} = 2 \times \alpha_N = \frac{4MG}{\xi c^2}$$

As claimed last week…