THE INFLATIONARY SCENARIO IN THE EARLY UNIVERSE

by

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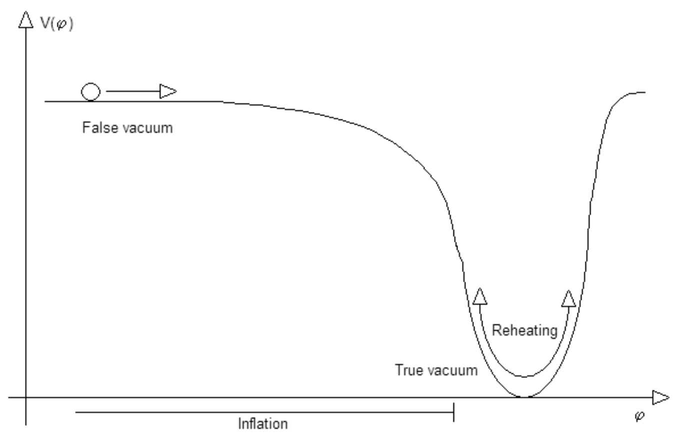

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Abstract

We present a review of the inflationary scenario. A description of the *Cosmological Standard Big Bang Model* (CSM) is given and it is explained how the problems of the CSM are solved by inflation, i.e., the horizon, flatness, large scale structures and monopole problems. In particular it is shown that inflation with a duration of 60 Hubble times implies an increase in the horizon distance d_H by 27 orders of magnitude, a decrease in the density parameter minus one $|\Omega - 1|$ by 54 orders of magnitude and a dilution of the concentration of monopoles by 78 orders of magnitude. Moreover, the *Slow Roll approximation* (SR), which is a specific model of inflation, is described. The parameters of this model, i.e., ϵ, η and ξ^2 can be directly compared with cosmological observations. Using the results from the Wilkinson Microwave Anisotropy Probe three-year data (WMAP3), it shall be argued that the SR approximation is a valid model for the early Universe.

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Contents

1 Introduction

The *Cosmological Standard Big Bang Model* (CSM) has since the discovery of the cosmological expansion and the *cosmological microwave background radiation* (CMB) been the leading theoretical framework for the evolution of the Universe. However, some serious problems have appeared. These are the horizon, flatness, large scale structures and monopole problems.

In the early 1980s a new framework was invented by [Guth, 1981] and [Linde, 1982]; the *inflationary scenario*. It proposed that from $10^{-36}s$ to $10^{-31}s$ after Big Bang, the Universe went through a period of exponential expansion. Inflation is inferred as an initial condition of the CSM model and solves the main problems of the CSM.

The exponential expansion could, according to [Linde, 1982], be caused by a scalar potential arising from a scalar field. In this report the theory of inflation is based on a single scalar field, the inflaton φ, and it is seen how the problems of the CSM are solved. Ever since the first framework of inflation was proposed, many variations have arisen. In this report the *Slow Roll approximation* (SR) is chosen, since it according to recent data, is a reliable approximation [Kinney et al., 2006]. We show how observations are used to test the SR. Even though, the inflationary scenario provides a nice framework for solving the problems of the CSM, some serious problems still remain to be answered.

It should be noted that in this report natural units are used, meaning that $c = \hbar = 1$, unless where otherwise mentioned.

2 Standard Big Bang Cosmology

In this section an overview of the CSM is presented. This report is based on the equations of this section and they form the basis of all other equations to be derived later. The CSM is a mathematical model based on the Einstein field equations Eq. (2.5), given that the *cosmological principle* can be used as an assumption. The cosmological principle states that the space-time in the Universe can be approximated to be homogeneous and isotropic. We shall in this report assume that this approximation holds for scales larger than $200Mpc$. There is somewhat different estimates of the size of these scales [Ryden, 2003, Lachieze-Rey et al., 1999]. Furthermore, it is assumed that the matter in the Universe can be described by simple thermodynamics; i.e, by a perfect fluid only depending on the density ρ and the pressure p [Kolb and Turner, 1999].

To be able to describe the evolution in time, the *cosmic standard time* t is defined. It is scaled so that $t_{Big\ Bang} = 0$ and $t_{today} = t_0$. The temperature T of the Universe is often used as an expression of time too.
The thermodynamical assumption ensures a relation between the temperature and the cosmic standard time. When the Universe is dominated by radiation, the relation between time and temperature is given by

$$T_{rad}(t) \simeq T_{planck} \left(\frac{t_{planck}}{t} \right)^{1/2} \simeq \frac{10^6}{t^{1/2}} \left[eV\ s^{1/2} \right] \tag{2.1}$$

where the Planck temperature is $T_{planck} = 1.2 \cdot 10^{28} eV$ and the Planck time is $t_{planck} = 5.39 \cdot 10^{-44} s$. Hence, a rule of thumb is that $t = 1s$ corresponds to $T_{rad} = 1MeV$, which according to [Bergstrom and Goobar, 1999] is the time where the Big Bang Nucleosynthesis, i.e., the time where atoms were formed, took place.

When describing the Universe we use the co-moving coordinates (t, r, θ, ϕ). This means that in a Universe, which obeys the cosmological principle, only a function of time, e.g., the scale factor (see page 5), is needed to describe an object's location, etc.

2.1 The Robertson Walker metric

The *Robertson Walker metric* (RW) is a spatially homogeneous and isotropic metric, which obeys the cosmological principle and the laws of thermodynamics. The derivation of the RW metric is beyond the scope of this text. We shall only state the general form in co-moving coordinates (t, r, θ, ϕ) [Weinberg, 1972].

$$d\tau^2 = dt^2 - \mathcal{R}^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + sin^2\theta d\phi^2\right)\right) \tag{2.2}$$

Where $d\tau$ is the *proper time* and is by definition an expression of the distance between two events in the space-time geometry [Olesen and Ambjrn, 2003]. $\mathcal{R}(t)$ is the *curvature scalar* (Sec. 2.2) and k is the *spatial curvature constant*, which denotes whether the Universe is positively curved ($k = +1$), negatively curved ($k = -1$) or flat ($k = 0$).

By introducing a new radial coordinate x and defining $r \equiv S_k(x)$ the RW metric can be rewritten [Ryden, 2003]

$$S_k(x) \equiv \begin{cases} sin(x) & k = +1 \\ x & k = 0 \\ \sinh(x) & k = -1 \end{cases} \tag{2.3}$$

$$\Downarrow$$

$$d\tau^2 = dt^2 - \mathcal{R}^2(t)\left(dx^2 + S_k^2(x)\left(d\theta^2 + sin^2\theta d\phi^2\right)\right) \tag{2.4}$$

This form is used when defining the proper distance in Sec. 3. The metrics Eqs. (2.2) and (2.4) do not look the same, but they still represent the same homogeneous and isotropic space, only with different choices of radial coordinate.

2.2 Deriving the Friedmann Equation

In this section the RW metric and the assumptions of homogeneity, isotropy and thermodynamics are used to derive the *Friedmann equation*. This equation describes the connection between the expansion of the Universe, the energy density and a possible cosmological constant. When dealing with the evolution of the Universe the Friedmann equation is the main equation.

The Einstein field equations for gravitation are a set of non-linear second order differential equations and allows a computation of the gravitational field from a given energy-momentum distribution. In tensor form the field equations are given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = -8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{2.5}$$

$G_{\mu\nu}$ is the *Einstein tensor*. It is symmetric, so that the order of the subscripts on T, R and g are of no importance. G is the well-known gravitational constant and Λ is the *cosmological constant*, which will be described in Sec. 2.3.
The subscripts μ and ν refer to the chosen co-moving coordinates. When doing calculations i and j represents the three co-moving spatial coordinates r, θ and ϕ and t is the

time coordinate. Thus, for the different tensors the time-time component is denoted by $_{tt}$, the space-space components are denoted by $_{ij}$ and the mixed space-time components are given by $_{it}$.

The quantity $g_{\mu\nu}$ is the *metric tensor.* It describes the relation between the different components of a given location in space. For instance, in Euclidean space the metric tensor is just the Kronecker delta matrix, which gives the well known x, y and z components. The metric tensor has the ability to manipulate the subscripts of other tensors via contraction or expantion of the given tensor.

The tensor $R_{\mu\nu}$ is the *Ricci tensor* and it is defined as the contraction of the Riemann-Christoffel curvature tensor: $R^{\lambda}_{\mu\lambda\nu} \equiv R_{\mu\nu}$. This tensor controls the geometry, i.e., for instance the curvature in the space-time. Contracting the Ricci tensor gives the curvature scalar $\mathcal{R}$. The Ricci tensor controls the rate of growth in the space, which in our case is the 4D space-time.

The *energy momentum tensor* $T_{\mu\nu}$ is sometimes called the stress energy tensor and describes, as indicated by the name, the energy and the momentum relations between the different components of the co-moving coordinates. The reason why it is sometimes called the stress energy tensor, is that $T_{\mu\nu}$ corresponds to the four-momentum across a surface - which is the stress on this surface. When assuming homogeneity and isotropy we have that $T_{tt} = \rho(t)$, $T_{ij} = p(t)g_{ij}$ and $T_{it} = 0$, where ρ is the energy density (Sec. 2.3) and p is the pressure, which is the stress on a given surface [Weinberg, 1972]. This clearly illustrates the properties of the energy momentum tensor.

Using rather simple tensor manipulations and the assumption that the energy momentum tensor $T_{\mu\nu}$ is described by a perfect fluid, it is possible to rewrite Eq. (2.5). The steps in rewriting the Einstein field equations from Eq. (2.5) to (2.6) are outlined in App. A.

$$R_{\mu\nu} = -8\pi G\left[\frac{1}{2}(\rho - p)g_{\mu\nu} + (p+\rho)g_{\mu\nu}U^{\nu}g_{\mu\nu}U^{\mu}\right] - \Lambda g_{\mu\nu} \tag{2.6}$$

The U^{μ}s and U^{ν}s are the four-velocities in space-time. It should be noted that ρ and p only depend on time, because of homogeneity and isotropy. This form of Einstein's fields equations is preferable, since it makes the derivation of the Friedmann equation easier. The components of the metric tensor $g_{\mu\nu}$ are the coefficients to the differential parts in the RW metric Eq. (2.2). As an example, the time-time part of the metric tensor is the coefficient to the dt^2-term of Eq. (2.2). The coefficients are

$$g_{tt} = -1 \quad ; \quad g_{it} = 0 \quad ; \quad g_{ij} = \mathcal{R}^2\tilde{g}_{ij} = \mathcal{R}^2 \begin{cases} \frac{1}{1-kr^2} & g_{rr} \\ r^2 & g_{\theta\theta} \\ r^2 sin^2\theta & g_{\phi\phi} \\ 0 & g_{i\neq j} \end{cases} \tag{2.7}$$

The change in signs compared to Eq. (2.2) is a result of the chosen diagonal $(-,+,+,+)$. This does not change the physics and is exclusively done to make the derivation in this section.

The Ricci tensor is connected to the affine connection $\Gamma^{\lambda}_{\mu\nu}$ (see App. B). Most of the components of the affine connection are equal to zero [Weinberg, 1972]. This implies that some of the Ricci tensor components equal zero. The non-vanishing components of the Ricci tensor are stated in Eqs. (2.8) and (2.9).
The components of the Ricci tensor involving time, that is, the time-time and space-time

components, are given by [Weinberg, 1972]

$$R_{tt} = \frac{3\ddot{\mathcal{R}}}{\mathcal{R}} \quad ; \quad R_{it} = 0 \tag{2.8}$$

where a dot represents differentiation with respect to time. The spatial part of the Ricci tensor is

$$\begin{aligned} R_{ij} &= \tilde{R}_{ij} - (\mathcal{R}\ddot{\mathcal{R}} + 2\dot{\mathcal{R}}^2)\tilde{g}_{ij} \\ &= -2k\tilde{g}_{ij} - (\mathcal{R}\ddot{\mathcal{R}} + 2\dot{\mathcal{R}}^2)\tilde{g}_{ij} \\ &= -\left(2k + \mathcal{R}\ddot{\mathcal{R}} + 2\dot{\mathcal{R}}^2\right)\tilde{g}_{ij} \end{aligned} \tag{2.9}$$

Here it is assumed that $\tilde{R}_{ij} = -2k\tilde{g}_{ij}$, which according to [Weinberg, 1972] is true for maximally symmetric spaces such as the RW metric.
Since the space-time part is zero, we will now focus on the R_{tt}-component and the spatial Ricci tensor.

To estimate the four-velocities in Eq. (2.6) the following definitions from [Weinberg, 1972] are used

$$U^t \equiv 1 \qquad ; \qquad U^i \equiv 0 \tag{2.10}$$

The second equation states the fact that co-moving coordinates are used. It shows that the material of the Universe is at rest in the co-moving coordinate system (r, ϕ, θ).
Using Eqs. (2.6), (2.7) and (2.10) another expression of the different parts of the Ricci tensor appears.

$$\begin{aligned} R_{tt} &= -8\pi G\left(\frac{1}{2}(\rho - p)(-1) + (p+\rho)(-1)1(-1)1\right) - \Lambda(-1) \\ &= -8\pi G\left(-\frac{1}{2}\rho + \frac{1}{2}p + p + \rho\right) + \Lambda \\ &= -8\pi G\left(\frac{1}{2}\rho + \frac{3}{2}p\right) + \Lambda \\ &= -4\pi G\,(\rho + 3p) + \Lambda \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} R_{ij} &= -8\pi G\left(\frac{1}{2}(\rho - p)\mathcal{R}^2\tilde{g}_{ij} + (p+\rho)\mathcal{R}^2\tilde{g}_{ij}0\mathcal{R}^2\tilde{g}_{ij}0\right) - \Lambda\mathcal{R}^2\tilde{g}_{ij} \\ &= \left(-4\pi G(\rho - p) - \Lambda\right)\mathcal{R}^2\tilde{g}_{ij} \end{aligned} \tag{2.12}$$

The next step is to eliminate the $\ddot{\mathcal{R}}$-parts in the non-vanishing components of the Ricci tensor. By comparing Eqs. (2.8) and (2.11) the *acceleration equation* is obtained

$$\begin{aligned} R_{tt} &= \frac{3\ddot{\mathcal{R}}}{\mathcal{R}} = -4\pi G\,(\rho + 3p) + \Lambda \\ &\Downarrow \\ \ddot{\mathcal{R}} &= -\frac{4\pi G}{3}\,(\rho + 3p)\,\mathcal{R} + \frac{\Lambda}{3}\mathcal{R} \end{aligned} \tag{2.13}$$

and by comparing Eqs. (2.9) and (2.12) gives

$$\begin{aligned} R_{ij} &= -\left(2k + \mathcal{R}\ddot{\mathcal{R}} + 2\dot{\mathcal{R}}^2\right)\tilde{g}_{ij} = \left(-4\pi G(\rho - p) - \Lambda\right)\mathcal{R}^2\tilde{g}_{ij} \\ &\Downarrow \\ 0 &= -2k - \mathcal{R}\ddot{\mathcal{R}} - 2\dot{\mathcal{R}}^2 + 4\pi G(\rho - p)\mathcal{R}^2 + \Lambda\mathcal{R}^2 \\ &\Downarrow \\ \ddot{\mathcal{R}} &= \frac{4\pi G(\rho - p)\mathcal{R}^2 + \Lambda\mathcal{R}^2 - 2k - 2\dot{\mathcal{R}}^2}{\mathcal{R}} \end{aligned} \tag{2.14}$$

Combining the acceleration equation and Eq. (2.14) leads to the Friedmann equation

$$\begin{aligned} -\frac{4\pi G}{3}(\rho + 3p)\mathcal{R} + \frac{\Lambda}{3}\mathcal{R} &= \frac{4\pi G(\rho - p)\mathcal{R}^2 + \Lambda\mathcal{R}^2 - 2k - 2\dot{\mathcal{R}}^2}{\mathcal{R}} \\ &\Downarrow \\ -4\pi G\left(\frac{1}{3}\rho + p - p + \rho\right)\mathcal{R} &= \Lambda\mathcal{R}\left(1 - \frac{1}{3}\right) - 2\frac{k}{\mathcal{R}} - 2\frac{\dot{\mathcal{R}}^2}{\mathcal{R}} \\ &\Downarrow \\ \left(\frac{\dot{\mathcal{R}}}{\mathcal{R}}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{k}{\mathcal{R}^2} + \frac{\Lambda}{3} \end{aligned} \tag{2.15}$$

When using the RW metric, $\mathcal{R}$ describes the curvature of space and depends on the era, i.e., matter radiation or Λ, in which it is described. In other words, the curvature scalar *scales* the metric and the evolution of the Universe. Therefore, when dealing with the CSM, $\mathcal{R}$ is named the *scale factor*. The scale factor is consequently a geometrical dimensionless factor depending on time, which describes the changes of the distances in the Universe as a function of time. The scale factor is normalized so that $\mathcal{R}(today) = \mathcal{R}(t_0) \equiv 1$.

The Hubble parameter H is defined through the scale factor

$$H(t)^2 \equiv \left(\frac{\dot{\mathcal{R}}(t)}{\mathcal{R}(t)}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{k}{\mathcal{R}(t)^2} + \frac{\Lambda}{3} \tag{2.16}$$

Apart from the Friedmann equation the *equation of energy conservation* is also needed, when describing the Universe via. CSM. This equation can be derived by assuming, among other things, that the energy momentum tensor is conserved.

The equation of energy conservation is given by [Bergstrom and Goobar, 1999]

$$\dot{p}\mathcal{R}^3 = \frac{d}{dt}\left(\mathcal{R}^3(\rho + p)\right) = 3\mathcal{R}^2\dot{\mathcal{R}}(\rho + p) + \mathcal{R}^3(\dot{\rho} + \dot{p}) \tag{2.17}$$

$$\Downarrow$$

$$0 = 3\frac{\dot{\mathcal{R}}}{\mathcal{R}}(\rho + p) + \dot{\rho} \tag{2.18}$$

This equation is also known as the *fluid equation* [Ryden, 2003]. It is derived in App. B, and it will in Sec. 2.3 be used to find a relation between the energy density and the scale factor.

We note that the redshift, z can be expressed as a function of the scale factor. The redshift, is defined as the ratio between the light's wavelength when it was emitted and when it was observed, i.e., $z + 1 = \frac{\lambda_{obs}}{\lambda_{em}}$. Thus, $z > 0$ indicates that the wavelength is enlarged, and thereby has been shifted to the red part of the optical spectrum. When $z < 0$ the wavelength has become smaller and the light is blueshifted.

The wavelength of a photon is given by $\lambda \sim 1/\mathbf{p}$, where $\mathbf{p}$ is the momentum of the photon. The momentum of the photon can be expressed as a function of the expansion of the Universe[Kolb and Turner, 1999]

$$\mathbf{p} \propto \frac{1}{\mathcal{R}(t)}$$

Combining this expression with the definition of the redshift gives

$$z + 1 = \frac{\mathcal{R}(t_{obs})}{\mathcal{R}(t_{em})} = \frac{1}{\mathcal{R}(t)} \tag{2.19}$$

The last equality sign appears when assuming that the light is observed today, i.e., $t_{obs} = t_0$, and leaving out the emission subscript. Thereby, the redshift is the relative reddening of the observed lightwaves, caused by the expansion of the Universe.

2.3 The Energy Density ρ and the Density Parameter Ω

The evolution of the Universe is complicated by the fact that it contains components with different equations of state. In this section approximations are done to get a feeling of the energy density of the Universe.
It is assumed that in general the equation of state can be written in the simple linear form

$$p = w\rho \tag{2.20}$$

which is true for "substances of cosmological importance" [Ryden, 2003]. The w is the dimensionless *equation of state parameter*. For any given energy-component carrying momentum, w is given by [Ryden, 2003]

$$w \approx \frac{\langle v^2 \rangle}{3c^2} \tag{2.21}$$

Where $\langle v^2 \rangle$ is the root mean square thermal velocity of the given component. This equation gives $w = 0$ for a gas of non-relativistic particles, and $w = \frac{1}{3}$ for a gas of relativistic particles. These examples are of particular interest, because both components exist in our Universe today. For simplicity, we shall refer to the component of the Universe that consists of non-relativistic particles as "matter", and the component that consists of photons and other relativistic particles as "radiation".

Many cosmologists believe that the Universe consists of a third component. This component is defined as one providing the Universe with a positive acceleration ($\ddot{\mathcal{R}} > 0$), and is in general referred to as the "dark energy". Thereby, Eq. (2.13) gives $p < -\frac{1}{3}\rho$, which causes $w < -\frac{1}{3}$. One such dark energy component is the cosmological constant. The cosmological constant results in a constant energy contribution, meaning that $\dot{\rho} = 0$, which, according to the fluid equation (2.18), implies that $p = -\rho \Rightarrow w = -1$.

Assuming that the Universe consists of matter, radiation and a cosmological constant, the total equation of state is given by

The form of energy	w
Radiation	$+\frac{1}{3}$
Matter	$\simeq 0$
The cosmological constant Λ	-1

Table 2.1: The general values of the equation of state parameter w for different kinds of energy.

$$p = \sum_w w\rho_w \tag{2.22}$$

where the w-parameter as mentioned takes the values given in Table 2.1.

As long as there is no interaction between the different energy-components, the fluid equation (2.18) must obtain for each component separately. Hence, the component with equation of state parameter w obeys the equation

$$0 = \dot{\rho_w} + 3\frac{\dot{\mathcal{R}}}{\mathcal{R}}(\rho_w + p_w) \tag{2.23}$$

$$\Downarrow$$

$$0 = \dot{\rho_w} + 3\frac{\dot{\mathcal{R}}}{\mathcal{R}}(1 + w)\rho_w \tag{2.24}$$

If it is assumed that w is constant and the normalization $\mathcal{R}(t_0) = 1$ is used then

$$\rho_w(\mathcal{R}) = \rho_{w,0}\mathcal{R}^{-3(1+w)} \tag{2.25}$$

The zero denotes that it is the energy density at present for the given component. Summarising the three different components we see that the total energy density of the Universe is

$$\rho = \frac{\rho_{m,0}}{\mathcal{R}^3} + \frac{\rho_{r,0}}{\mathcal{R}^4} + \rho_{\Lambda,0} \tag{2.26}$$

where m and r indicates the matter and radiation contributions respectively. This equation gives that in the early Universe, when $\mathcal{R} \ll 1$, ρ was dominated by radiation. Later it was dominated mostly by matter, and at some point the energy density will be dominated by the energy corresponding to the cosmological constant.

When fitting the various parameters to observations, ρ is compared with the *critical density* ρ_c, which is the energy density that makes the Universe perfectly flat. This comparism is done because the number ρ alone does not provide any qualitative information. The *density parameter* Ω is therefore introduced

$$\Omega \equiv \frac{\rho(t)}{\rho_c} \tag{2.27}$$

From the Friedmann equation Eq. 2.16, the critical density corresponds to $\rho_c = \frac{3H^2}{8\pi G}$ by setting $k = 0$ and ignorring the Λ term, since this term does not have any influence in the early Universe. We could for instance estimate the size of Λ at *last scattering*, which is when the Universe became transparent to photons at $z_{ls} \simeq 1100$. Combining Eqs. (2.16), (2.19) and (2.26) gives that Λ should at least be of the order $\mathcal{R}_{ls}^{-2} \simeq 10^6$

to have influence on ρ_c. Since recent data indicates that $\Lambda \ll 10^6$, this is not the case [Spergel et al., 2006]. Using the definition of ρ_c and Ω it follows from Eq. (2.16) that

$$\frac{k}{\mathcal{R}^2} = \frac{8\pi G}{3}\rho_c(\Omega - 1) = H^2(\Omega - 1) \tag{2.28}$$

Hence, $\Omega > 1$, $\Omega = 1$ and $\Omega < 1$ correspond to $k = +1$, $k = 0$ and $k = -1$ respectively. The Eq. (2.28) has a simple intuitive appeal. A very large Ω gives a huge amount of energy, which counteracts the expansion and makes the Universe closed. Similarly, if Ω is very small, there is very little counteraction to expansion, and the Universe is open.

Since ρ_c is given in terms of the Hubble constant, and since Ω could in principle be determined by the mass density in the Universe, Eq. (2.28) allows a determination of the sign of k, and hence the fate of the Universe.

3 Problems in Standard Big Bang Cosmology

In this section we will present the four major problems which makes the CSM a defective model for the very early Universe. To describe the problems, we shall introduce the *proper distance* d_p, which is one of the distance notations in the CSM. Using the RW metric, assuming that the Universe is flat and if time is instantly frozen, i.e., $dt^2 = 0$, the *proper distance* is defined as [Ryden, 2003]

$$d_p(x,t) = \mathcal{R}(t)\int_0^x dx = \mathcal{R}(t)\int_{t_e}^{t}\frac{1}{\mathcal{R}(t')}dt' \tag{3.1}$$

where x is the radial co-moving coordinate and t_e is the time at which the light was emitted. The second equality sign is obtained by considering the way light travels toward us. Light travels along the metric's zero geodesics, i.e., $d\tau = 0$. Since the angles in the RW metric by definition are constant along a zero geodesic, one gets that $dt^2 = \mathcal{R}(t)^2 dx^2$, which by simple manipulation gives the second equality sign. The proper distance is used to quantify the horizon problem.

3.1 The Horizon Problem

Mesurements of the CMB show that the temperature fluctuations $\frac{\Delta T}{T} = 2 \cdot 10^{-5}$ at 10 degrees angular scales [Bergstrom and Goobar, 1999]. That the fluctuations are that small is a problem because, as we will show in this section, all regions in the Universe could not initially have been causally connected. If the regions have not at some point been in causal contact, it would be impossible for them to gain the same temperature with such a high precision.

For instance, from the proper distance defined in Eq. (3.1) the *horizon distance* d_H can be defined. The horizon distance is the proper distance which light has traveled in a given time. This then describes the horizon of the visible Universe for an observer "located" at t_0 receiving the light emitted at t_H. Because of lack of time the light from distances greater than x_H has not arrived yet. Thereby

$$\begin{aligned} d_H &= d_p(t_0) = \mathcal{R}(t_0)\int_0^{x_H} dx \\ &= \mathcal{R}(t_0)\int_{t_H}^{t_0}\frac{1}{\mathcal{R}(t)}dt = \int_{t_H}^{t_0}\frac{1}{\mathcal{R}(t)}dt \end{aligned} \tag{3.2}$$

Furthermore, assuming that the equation of state of the Universe is $p = w\rho$, as in Sec. 2.3, the Universe is flat and ρ is dominated by a single component, and combining this with Eqs. (2.16) and (2.25), gives

$$\mathcal{R}(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} \tag{3.3}$$

This expression for $\mathcal{R}(t)$ is derived in App. C and gives that

$$\begin{aligned} d_H &= \int_{t_H}^{t_0} \frac{1}{\mathcal{R}(t)} dt = \int_{t_H}^{t_0} \left(\frac{t}{t_0}\right)^{-\frac{2}{3(1+w)}} dt \\ &= \frac{3(1+w)}{1+3w} t_0 \left[\left(\frac{t}{t_0}\right)^{\frac{1+3w}{3(1+w)}}\right]_{t_H}^{t_0} \\ &= \begin{cases} 3t_0 \left(1 - \left(\frac{t_H}{t_0}\right)^{1/3}\right) & \text{; matter-dominated} \\ 2t_0 \left(1 - \left(\frac{t_H}{t_0}\right)^{1/2}\right) & \text{; radiation-dominated} \end{cases} \end{aligned} \tag{3.4}$$

Letting $t_H \ll t_0$ gives that the time between emission and observation is large and that the ratios in Eq. (3.4) can be approximated to zero. Hence, for matter- and radiation-domination the particle horizon distance is finite! If $t_H = 0$, the light was emitted at the same time as the Big Bang, which corresponds to the largest possible horizon distance.

The d_H is the size of the observable Universe inside which a particle is causally connected with all other particles. Material outside such a volume is therefore not causally connected to the particle from which the horizon distance is measured. If this constant distance is smaller than the size of the Universe, not all particles are causally connected. This creates the horizon problem because of the isotropy of the CMB. Furthermore, the usage of thermodynamics in the CSM is limited to this horizon, and one cannot state that this holds for the rest of the Universe, which was assumed in Sec. 2.2 when deriving the Friedmann equation.

To get an idea of the size of the problem one might ask the question: How large a fraction of the Universe is causally connected? This fraction could be given by $\frac{d_H}{\mathcal{R}}$. From Eq. (3.3) $\mathcal{R} \propto t^{2/3}$ and $\mathcal{R} \propto t^{1/2}$ for matter- and radiation-domination respectively. Letting t_0 in Eq. (3.4) represent the time t in the given eras, this makes the fraction evolve as:

$$\frac{d_H}{\mathcal{R}} \propto \begin{cases} t^{1/3} & \text{; matter-dominated} \\ t^{1/2} & \text{; radiation-dominated} \end{cases}$$

Thus as time passes, more and more particles become causally connected. This means that in the distant past the horizon problem was even more pronounced.

To describe the horizon problem more quantitatively one can use the total entropy of the Universe as a measure of the causally connected regions. As mentioned earlier one of the assumptions when dealing with the CSM is that the Universe is described by simpel thermodynamics. The total entropy S in the differential form of the first law of thermodynamics is defined as

$$dS = \frac{d(\rho V) + p dV}{T} = \frac{d\left((\rho + p)V\right) - V dp}{T} \tag{3.5}$$

where V is the volume considered and T is the temperature. If we assume that the volume is the entire Universe, so that $V = \Re^3$, and that the energy is conserved Eq. (2.17), then the entropy is conserved in thermal equilibrium, that is $dS = 0$. Rewriting Eq. (3.5) and integrating over dS gives the total entropy S (see App. D).

$$S = \frac{V}{T}(\rho + p) \tag{3.6}$$

Using Bose Einstein statistics on the radiation-dominated era, an expression for the energy density can be derived, and using from Sec. 2.3 that relativistic particles is governed by the equation of state $\rho = p/3$, gives

$$\rho_{rad}(T) = \frac{\pi^2}{30} g_{eff}(T) T^4 \tag{3.7}$$

$$p_{rad}(T) = \frac{1}{3}\rho_{rad}(T) = \frac{\pi^2}{90} g_{eff}(T) T^4 \tag{3.8}$$

where g_{eff} is the total number of effective degrees of freedom [Kolb and Turner, 1999].

Combining these expressions for ρ and p with the expression for the total entropy and defining the *entropy density* $s = S/V$, we get for radiation-domination

$$s_{rad} = \frac{2\pi^2}{45} g^s_{eff} T^3 \tag{3.9}$$

When matter dominates the Universe, the entropy density is defined as $s_{mat} = s(today)(1+z)^3 = 2970(1+z)^3\ cm^{-3}$, where $s(today)$ is calculated from equation (3.9) with the values $g^s_{eff}(today) = 3.91$, $g_{eff}(today) = 3.36$ and $T(today) = 2.75K$ [Kolb and Turner, 1999]. In the equation above

$$g^s_{eff} \equiv \sum_{i=bosons} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{j=fermions} g_j \left(\frac{T_j}{T}\right)^3 \tag{3.10}$$

The g_{eff} is defined in the same way as g^s_{eff}. The only difference being that the exponents for g_{eff} is four instead of three.

The Hubble parameter goes as $\frac{1}{2t}$ for radiation-domination, which combined with Eq. (3.7) gives

$$t \propto \frac{2}{3}\left(\frac{8\pi G}{3}\rho_{rad}(T)\right)^{-1/2} = \frac{2\sqrt{30}}{3\pi}\left(\frac{8\pi}{3}\right)^{-\frac{1}{2}} \frac{m_{pl}}{\sqrt{g_{eff}} T^2} \simeq 0.4 \frac{m_{pl}}{\sqrt{g_{eff}} T^2} \tag{3.11}$$

Here the general notation of the Planck mass $m_{pl} = G^{-1/2} \simeq 10^{19} GeV$ is used. If s_{rad} is multiplied with the volume of the Universe inside the horizon, i.e., the approximation of Eq. (3.4), the total entropy is given by

$$\begin{aligned} S^{rad}_{Horizon} &= \frac{4\pi}{3} d_H^3 s_{rad} = \frac{32\pi}{3} t^3 s_{rad} & (3.12)\\ &= \frac{32\pi}{3}\left(0.4 \frac{m_{pl}}{\sqrt{g_{eff}} T^2}\right)^3 \frac{2\pi^2}{45} g^s_{eff} T^3 \\ &\simeq 0.94 \frac{g^s_{eff}}{{g_{eff}}^{3/2}} \left(\frac{m_{pl}}{T}\right)^3 \\ &\simeq 0.94 {g_{eff}}^{-1/2} \left(\frac{m_{pl}}{T}\right)^3 \quad ; \quad \text{for } t < 1s & (3.13) \end{aligned}$$

The last equality sign appears when assuming that $g^s_{eff} \simeq g_{eff}$. This approximation is valid when the temperature ratios in the definitions of g^s_{eff} and g_{eff} are close to unity, i.e., all the particle species have common temperatures. This is the case as long as neither the neutrinos are decoupled nor the electrons annihilate. At temperatures $T \sim 1MeV$ neutrinos decouple and electrons annihilate, so according to Eq. (2.1), Eq. (3.13) holds for $t < 1s$.

Combining Eqs. (3.9) and (3.12) with the fact that the horizon distance for matter-domination can be written as $d_{H,mat} = \frac{2}{H_0}(1+z)^{-3/2}$ [Kolb and Turner, 1999], gives an estimate of the total entropy in the matter-dominated era

$$S^{mat}_{Horizon} = \frac{4\pi}{3} d^3_H s_{rad} = \frac{4\pi}{3}\left(\frac{2}{H_0}\right)^3 (1+z)^{-9/2} \frac{2\pi^2}{45} g^s_{eff} T^3 \tag{3.14}$$

where H_0 is the present Hubble time. Correcting for natural units so that $d_H = \frac{c}{H_0} \simeq 10^{28} cm$ and using that the temperature goes as $T_{CMB,0}/\mathcal{R}$ gives that

$$S^{mat}_{Horizon} \simeq 10^{86}(1+z)^{-3/2} \tag{3.15}$$

We thus see that the total entropy, and thereby the number of causally disconnected regions, becomes smaller as z grows, that is $\mathcal{R} \to 0$. At last scattering ($z_{ls} \simeq 1100$) the entropy was $\sim 10^{81}$; that is 10^5 times smaller than the entropy at present. That 10^5 regions which were causally disconnected when light was emitted have the same temperature today is the horizon problem formulated in the context of the total entropy of the Universe.

3.2 The Flatness Problem

The flatness problem arises because ρ is very close to ρ_c [Ryden, 2003]. Using Eqs. (2.16), (2.26) and (2.27) we see that.

$$\begin{aligned} \Omega - 1 &= \frac{k}{\mathcal{R}^2 H^2} - \frac{\Lambda}{3H^2} \\ &= \frac{k - \frac{\mathcal{R}^2\Lambda}{3}}{\frac{8\pi G}{3}\left(\frac{\rho_{m,0}}{\mathcal{R}} + \frac{\rho_{r,0}}{\mathcal{R}^2} + \mathcal{R}^2\rho_{\Lambda,0}\right) - k + \frac{\mathcal{R}^2\Lambda}{3}} \end{aligned} \tag{3.16}$$

When leaving out the Λ-term this gives

$$\Omega - 1 = \frac{k}{\frac{8\pi G}{3}\left(\frac{\rho_{m,0}}{\mathcal{R}} + \frac{\rho_{r,0}}{\mathcal{R}^2}\right) - k} \propto \begin{cases} \mathcal{R} & ; \text{matter-dominated} \\ \mathcal{R}^2 & ; \text{radiation-dominated} \end{cases} \tag{3.17}$$

Thus, it is seen that $\Omega - 1$ goes as $\mathcal{R}(t)$ and $\mathcal{R}^2(t)$ when the Universe is matter and radiation dominated respectively. Hence, in the early radiation dominated Universe, where $\mathcal{R}$ was small, the difference between 1 and Ω was very small.

Observations show that today $\Omega = 1 \pm 0.2$ [Ryden, 2003]. From Eq. (3.17) it follows that in the early Universe $|\Omega - 1| \simeq 10^4(z+1)^{-2}$ [Kolb and Turner, 1999]. Estimating the value of the density parameter at the time of Big Bang Nucleosynthesis, i.e., when the temperature was $1MeV$, gives

$$\Omega_{BBN} = 1 \pm 10^4 \left(\frac{T_{BBN}}{T_{CMB,0}}\right)^{-2} = 1 \pm 10^4 \left(\frac{10^6 eV}{2.35 \cdot 10^{-4} eV}\right)^{-2} \simeq 1 \pm 6 \cdot 10^{-16} \tag{3.18}$$

where it is used that $T_{CMB,0} = 2.35 \cdot 10^{-4} eV$. That the initial value of the density parameter is one with such a high accuracy is the flatness problem.

3.3 Large Scale Structures

A third problem is that CSM does not explain the excistense of large scale structures in the Universe, e.g., galaxies, galaxy clusters, filaments and voids. When deriving the Friedmann equation it is assumed that the Universe is homogeneous and isotropic. To create the observed large scale structures there must have been a mechanism causing the Universe to be anisotropic on small scales, so that gravitational infall could have created the observed structers. If such a break did not occur, no large scale structures would have been formed, because of the assumption of homogeneity and isotropy in CSM.

3.4 The Monopole Problem

When the Universe had an age of $\sim 10^{-12}s$, there was enough energy to fuse the electromagnetic force (EM) and the weak nuclear force into one force. For earlier times physicists believe that also the strong nuclear force was fused with the other two forces. This should have happened when the temperature reached $10^{28}K$, i.e., when the Universe was around $10^{-36}s$ old [Ryden, 2003]. This fusion of the EM, the weak and the strong forces is in literature referred to as the Grand Unified Theory (GUT).

If this Grand Unification has happened, the phase transition followed by the breaking of symmetry would have caused a series of *topological defects*. These topological defects are, among others, cosmic strings and point-like defects, which behave as magnetic monopoles. These would have dominated the Universe completely because of their high masses [Kolb and Turner, 1999].

The fact that there will be an overweight of magnetic monopoles after symmetry breaking, together with the fact that magnetic monopoles are not observed, is another problem with the CSM. How can it be that we do not see the topological defects today when it seems that they dominated the very early Universe? How did they disappear, if they have existed at all?

These questions as well as the questions mentioned above have to be answered to get a precise model for the Universe at all times. The model that might give us the answers is the inflationary scenario.

4 The Inflationary Scenario

In general the CSM has many successful features and gives a nice framework for discussing the rather few astronomical observations governing the very early Universe. However, as described in Sec. 3 certain problems with the initial conditions emerge from the model. In search of the best physical model of the Universe the CSM has to be supplied with further theory to ensure the correct initial conditions. Today the best supplement to the CSM is the *inflationary scenario*.

The basic idea of inflation is proposed by [Guth, 1981] and is extended and corrected by [Linde, 1982]. They argue that shortly after the Big Bang, an extreme expansion, which cause a positive acceleration, blow up the Universe. After inflation the Universe enter the radiation-dominated epoch described by the CSM. Within this broad framework, many specific models for inflation has been proposed. In this report models with "normal" gravity, i.e., general relativity, and which describe the vacuum by a single scalar field φ, the *inflaton*, are considered. It is assumed that the quantum fluctuations in the scalar field are negligible compared to the classical part of the field. According to [Kolb and Turner, 1999] this is a fairly good approximation. Hence, working with the scalar field

$$\varphi(t) = \varphi_{Classic} + \varphi_{Quantum} \simeq \varphi_{Classic} \tag{4.1}$$

The potential arising from this scalar field will be referred to as $V(\varphi)$.

The main feature of inflation is the transition from a *false* vacuum where inflation takes place, to the *true* vacuum which corresponds to the end of inflation. Often this transition results in some field-oscillations. The duration of these oscillation is called the epoch of *reheating*. We will not go into details about reheating, only mentioning that in this process particles are created and that the temperature rises to $\sim 10^{15} GeV$. A potential with such features could be a typical slow roll potential, as in Fig. 6.1 page 19.

During inflation the energy density ρ and the pressure p are dominated by the scalar field φ. When assuming that φ is spatially homogeneous and using a Lagrangian on the form $\mathcal{L} = \partial^\mu\varphi\partial_\mu\varphi/2 - V(\varphi)$, the ρ and p are given by

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \qquad , \qquad p_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi) \tag{4.2}$$

When using Eq. (4.2) and the conservation of energy-momentum, ($T^{\mu\nu}{}_{;\nu} = 0$), the *equation of motion* of the inflaton is given by

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0 \tag{4.3}$$

where H is the Hubble parameter defined in Eq. (2.16). A prime denotes $d/d\varphi$. For derivation of (4.2) and (4.3) see App. B.

To understand Eq. (4.3) more intuitively, the inflaton can be compared with a ball rolling down a hill. In this case; $\ddot{\varphi}$ is the acceleration, $\dot{\varphi}$ is the velocity with the friction term $3H$ (often referred to as the Hubble-friction) and $V'(\varphi)$ corresponds to the variation of the potential energy as the ball rolls.

Using the RW metric as the flat ($k = 0$) background metric without any contribution from the cosmological constant, the Friedmann equation, Eq. (2.16), is given by

$$H^2 = \left(\frac{\dot{\mathcal{R}}}{\mathcal{R}}\right)^2 = \frac{8\pi G}{3}\rho = \frac{8\pi}{3m_{pl}^2}[V(\varphi) + \frac{1}{2}\dot{\varphi}^2] \tag{4.4}$$

The last equality sign is obtained by inserting the expression for ρ, i.e., Eq. (4.2). Furthermore, when using Eq. (4.2) the acceleration equation Eq. (2.13) reduces to:

$$\begin{aligned}\left(\frac{\ddot{\mathcal{R}}}{\mathcal{R}}\right) &= -\frac{4\pi G}{3}\left(\frac{1}{2}\dot{\varphi}^2 + V + \frac{3}{2}\dot{\varphi}^2 - 3V\right) \\ &= \frac{8\pi}{3m_{pl}^2}(V(\varphi) - \dot{\varphi}^2)\end{aligned} \tag{4.5}$$

Thereby, the equation of energy conservation, Eq. (2.17), of the inflaton is

$$\begin{aligned} 0 &= -\dot{p}_\varphi \mathcal{R}^3 + 3\mathcal{R}^2\dot{\mathcal{R}}(\rho_\varphi + p_\varphi) + \mathcal{R}^3(\dot{\rho}_\varphi + \dot{p}_\varphi) \\ &= \dot{\rho_\varphi} + 3\frac{\dot{\mathcal{R}}}{\mathcal{R}}(\rho_\varphi + p_\varphi) \end{aligned} \tag{4.6}$$

In the inflation era the Universe is dominated by vacuum energy ($\rho = -p$). Such a Universe is called a *de Sitter* Universe [Kolb and Turner, 1999]. From Eq. (4.2) it is seen that this can be accomplished by assuming that $\dot{\varphi}^2 \ll V$. Assuming that φ is monotone in time, H can be expressed as a function of φ only.

$$H = \left(\frac{\dot{\mathcal{R}}}{\mathcal{R}}\right) = \sqrt{\frac{8\pi}{3m_{pl}^2}V(\varphi)} \tag{4.7}$$

$$\Downarrow$$

$$H' = \left(\frac{8\pi}{3m_{pl}^2}\right)^{1/2} \frac{1}{2}\frac{V'(\varphi)}{V(\varphi)^{1/2}} \tag{4.8}$$

Setting the potential to be constant in time, implies that H is constant, and the solution to Eq. (4.7) becomes

$$\mathcal{R} \propto e^{Ht} \tag{4.9}$$

which is the exponential expansion needed to solve the problems in CSM, see Sec. 5. From the assumption that $\dot{\varphi}^2 \ll V$ the equation of motion Eq. (4.3) can be re-expressed.

$$\begin{aligned} \dot{\varphi} &= -\frac{\ddot{\varphi} + V'(\varphi)}{3H} \\ &= -\frac{\ddot{\varphi} + V'(\varphi)}{3\sqrt{K^2V(\varphi)}} \\ &= -\frac{2}{3K^2}\left(\frac{1}{2}K\frac{\ddot{\varphi}}{V(\varphi)^{1/2}} + \frac{1}{2}K\frac{V'(\varphi)}{V(\varphi)^{1/2}}\right) \\ &= -\frac{m_{pl}^2}{4\pi}\left(\frac{1}{2}K\frac{\ddot{\varphi}}{V(\varphi)^{1/2}} + H'(\varphi)\right) \end{aligned} \tag{4.10}$$

Here K is for simplicity defined as

$$K \equiv \left(\frac{8\pi}{3m_{pl}^2}\right)^{1/2} \tag{4.11}$$

Furthermore, we can rewrite the acceleration equation Eq. (4.5) once more by substituting the expressions for V and $\dot{\varphi}$. This gives that

$$
\begin{aligned}
\frac{\ddot{\mathcal{R}}}{\mathcal{R}} &= K^2\left(H^2K^{-2} - \frac{m_{pl}^4}{16\pi^2}\left(\frac{1}{2}K\frac{\ddot{\varphi}}{V(\varphi)^{1/2}} + H'\right)^2\right) \\
&= H^2 - \frac{m_{pl}^2}{6\pi}H'^2 - \frac{m_{pl}^2}{6\pi}\left(\frac{(K\ddot{\varphi})^2}{4V(\varphi)} + K^2\ddot{\varphi}\frac{H'}{H}\right) \\
&= H^2 - \frac{m_{pl}^2}{6\pi}H'^2 - F(\ddot{\varphi}) \\
&= H^2\left(1 - \frac{m_{pl}^2}{6\pi}\left(\frac{H'}{H}\right)^2\right) - F(\ddot{\varphi}) \\
&= H^2\left(1 - \frac{2}{3}\,\epsilon\right) - F(\ddot{\varphi}) \qquad (4.12)
\end{aligned}
$$

where

$$\epsilon \equiv \frac{m_{pl}^2}{4\pi}\left(\frac{H'}{H}\right)^2 \qquad (4.13)$$

and

$$F(\ddot{\varphi}) \equiv \frac{m_{pl}^2}{6\pi}\left(\frac{(K\ddot{\varphi})^2}{4V(\varphi)} + K^2\ddot{\varphi}\frac{H'}{H}\right) \qquad (4.14)$$

The definition of $F(\ddot{\varphi})$ is made for simplicity. The ϵ is defined since it is one of the SR parameters, which can be compared to observations (Sec. 6.1).

In order to satisfy the assumption $\ddot{\mathcal{R}} > 0$ during inflation, we need $\epsilon < 3/2$ and $F(\ddot{\varphi})$ to be insignificant. Our calculations deviate a little from for instance [Kinney, 2002], since they get that $\epsilon < 1$ Because $1 \simeq 3/2$ this does not change anything crucial.

As a consequence of Eq. (4.9) the evolution of the scale factor from the beginning until the end of the inflation era can be written as

$$\mathcal{R} \propto \exp\left(\int_{t_{initial}}^{t_{final}} H dt\right) \qquad (4.15)$$

The number of e-foldings, N, is a measure of the duration of inflation, and is defined as [Kinney, 2002]

$$N \equiv \int_{t_i}^{t_f} H dt = \int_{\varphi_i}^{\varphi_f} \frac{H}{\dot{\varphi}} d\varphi \qquad (4.16)$$

When [Kinney et al., 2006] analyze the latest data and do model calculations for comparisons, they use that $46 < N < 60$. We shall in the rest of this text use the value $N = 60$, since this is commen consensus [Hansen and Kunz, 2002, Kolb and Turner, 1999, Lyth and Riotto, 1999] and solves the problems of CSM as shown in the next section.

5 Solving the Problems of the Standard Model

To see how inflation solves the problems of the CSM, it is imagined that inflation is switched on instantaneously at a time t_i and switched off at a time t_f. It is proposed that the Universe prior to the exponential growth is radiation-dominated and return to this former state after inflation (the exact transition during and immediately after reheating is unknown, there may have been a brief period of matter-domination [Linde, 2005]). As mentioned, the Hubble parameter is constant during inflation. We shall indicate this constant by H_i, which is the value of H when inflation starts. Using Eq. (3.3) and Eq. (4.15) the scale factor is then given by

$$\mathcal{R} = \begin{cases} \mathcal{R}_i(t/t_i)^{1/2} & t < t_i \\ \mathcal{R}_i e^{H_i(t-t_i)} & t_i < t < t_f \\ \mathcal{R}_i e^{H_i(t_f-t_i)}(t/t_f)^{1/2} & t > t_f \end{cases} \tag{5.1}$$

Thus, during inflation the scale factor increases by a factor

$$\frac{\mathcal{R}_f}{\mathcal{R}_i} = e^{H_i(t_f-t_i)} = e^N \tag{5.2}$$

If inflation lasted for a long time, compared to the Hubble time during inflation, H_i^{-1}, then N was large, and the growth in the scale factor during inflation was enormous. Exactly this is the main reason why inflation solves the CSM problems.

5.1 The Horizon problem

We shall now see how inflation solves the horizon problem. The horizon distance at any time t is given by Eq. (3.1). At the beginning of inflation d_H is

$$d_H(t_i) = \mathcal{R}_i \int_0^{t_i} \frac{dt}{\mathcal{R}_i(t/t_i)^{1/2}} = 2t_i \ , \tag{5.3}$$

and at the end of inflation d_H is

$$d_H(t_f) = \mathcal{R}_i e^N \left(\int_0^{t_i} \frac{dt}{\mathcal{R}_i(t/t_i)^{1/2}} + \int_{t_i}^{t_f} \frac{dt}{\mathcal{R}_i exp[H_i(t-t_i)]} \right) \tag{5.4}$$

For a large amount of e-foldings Eq. (5.4) reduces to

$$d_H(t_f) = e^N(2t_i + H_i{}^{-1}) = 3e^N t_i \tag{5.5}$$

A specific example shows how the exponential growth solves the problem. In the calculations c is included to correct for natural units. Assuming that inflation starts at $t_i \sim 10^{-36}s$, and thereby assuming that the Hubble-parameter is $H_i \sim 10^{36}s^{-1}$, the size of the horizon at the beginning of inflation is

$$d_H(t_i) = 2ct_i \approx 2(3 \cdot 10^{10} \cdot 10^{-36})cm = 6 \cdot 10^{-26}cm \ , \tag{5.6}$$

The size of the horizon immediately after inflation is then

$$d_H(t_f) = 3c \cdot e^N t_i \approx 3(3 \cdot 10^{10} \cdot e^{60} \cdot 10^{-36}) \approx 10cm \tag{5.7}$$

Hence, during inflation the horizon distance is increased by 27 orders of magnitude. So even though $10cm$ does not seem as much in a cosmological context the expansion is enormous.

The huge expansion ensures that the entire last scattering surface could have been in causal contact before the epoch of inflation, and the expansion thus gives the isotropy that we observe today.

As in Sec. 3.1, the solution to the horizon problem may also be described in the context of entropy. We want to show that the $\sim 10^{81}$ regions at last scattering, evolve from a causally connected area.

From the inflationary scenario it is known that reheating increases the temperature with several orders of magnitude to $\sim 10^{15} GeV$. Since the entropy is constant during inflation, the entropy at the end of inflation is given by $S_f = e^{3N} S_i$. The volume equals $\mathcal{R}^3$ and is therefore blown up with a factor e^{3N}. Thus even a small amount of causally connected regions before inflation become $2 \cdot 10^{78}$ times larger for $N = 60$.

As an example, a region before inflation of $10^{-23} cm$ has an entropy of roughly 10^{14} [Kolb and Turner, 1999] and is turned into a region with an entropy of 10^{92}. Since the entropy is a constantly increasing factor, the estimate of the causally connected regions today will easily fit inside the originally causally connected region. Thereby the inflationary model also solves the horizon problem when looking at it from the entropy point of view.

5.2 The Flatness problem

As mentioned, the Hubble parameter is constant during inflatio. This combined with Eq. (3.17) gives an expression of the $|\Omega(t) - 1|$-part in the inflationary era

$$|\Omega(t) - 1| = \frac{k}{\mathcal{R}^2 H^2} \propto \mathcal{R}^{-2} \tag{5.8}$$

It is assumed that k is not exactly zero and that the Λ-contribution is ignored. The scale factor in the period of inflation is exponential, thereby, at the end of inflation

$$|\Omega(t) - 1| \propto e^{-2H_i t} \tag{5.9}$$

Thus, comparing the density parameter at the beginning and at the end of inflation gives

$$|\Omega(t_f) - 1| \propto e^{-2H_i\left(\frac{N}{H_i}+t_i\right)} = e^{-2N}|\Omega(t_i) - 1| \tag{5.10}$$

Since $t = t_f = N/H_i + t_i$. This gives that

$$|\Omega(t_f) - 1| \sim e^{-2N}|\Omega(t_i) - 1| \sim e^{-120}|\Omega(t_i) - 1| \sim 10^{-54}|\Omega(t_i) - 1| \tag{5.11}$$

Even if the density parameter at the beginning of inflation is huge or very small, the inflation mechanism lower the density with 54 orders of magnitude. The fact that $\Omega = 1 \pm 6 \cdot 10^{-16}$ one second after Big Bang is now easily accomplished since, using Eqs. (3.3) and (3.17), the density parameter for radiation-dominated eras evolves as

$$|\Omega - 1| \propto t^{\frac{2(1+3w)}{3(1+w)}} = t \tag{5.12}$$

Even if the Universe at t_i is not flat at all ($\Omega \gg 1$), 60 e-foldings of inflation will make the Universe extremely flat, i.e., $\Omega = 1$. Hence the inflationary scenario solves the flatness problem.

5.3 Large Scale Structures

The problem of density fluctuations which evolve to large scale structures, is also solved within the inflationary scenario. As mentioned, the inflaton field has quantum effects, which are much smaller than the classical part of the field. These quantum fluctuations are nevertheless under inflation moved beyond the Hubble radius because of the rapid expansion of the Universe. As the fluctuations pass the horizon, they are "freezed out", meaning that they become classical. This results in a conversion from quantum energy perturbations into classical matter perturbations. As time passes the Hubble horizon "catches up" with the matter density fluctuations, and they re-enter the observable Universe. As CSM takes over after inflation, the grains for large scale structure formation and gravitational infall are now seeded because of the huge expansion. Thus, inflation gives an explanation of the large scale structure problem [Kolb and Turner, 1999].

5.4 The Monopole problem

As mentioned when solving the horizon problem, the volume of the Universe is blown up by a factor of e^{3N} during inflation. This effect solves the monopole problem.

If it is assumed that the topological defects did in fact exist, it is seen that after inflation the number density is diluted by a factor of e^{3N}. This means that any concentration of monopoles or cosmic strings after inflation becomes $\sim 10^{78}$ times smaller (for $N = 60$). Therefore, if the topological defects were created it would be almost impossible to observe any of them today. Thus, the monopole problem is solved by presuming the inflationary scenario.

6 The Slow Roll Approximation

In this section we will make the *Slow Roll approximation* (SR) to the inflationary scenario. The SR is chosen since it is comparable with observations of the CMB. The *SR parameters* are directly connected to the observable power spectra and spectral indices, and thereby to CMB observations (Sec. 6.1). Every inflation model must fit the CMB-data in order to be accepted. In Sec. 7 we shall describe how [Kinney et al., 2006] recently did the comparisons with CMB, and use their results to determine whether the SR approximations is a good model.

The SR demands a rather flat potential so that the rolling ball, see Sec. 4, spends enough time on the flat part of the potential V. This ensures that the duration of inflation is long enough to solve the CSM problems.

A typical SR potential has a false and a true vacuum. The false vacuum corresponds to the part of the potential where the ball is rolling slowly and inflation takes place. At some point the ball rolls into the true vacuum, which corresponds to the end of inflation and the beginning of the epoch where the CSM becomes valid. A SR potential could take the form shown in Fig. 6.1. Before the ball ends up at $V(\varphi) = 0$, it will oscillate in the true vacuum. This oscillation is the reheating as described in Sec. 4.

The actual SR is given by the assumption that the evolution of the field is dominated by a pull from the expansion of the Universe, which corresponds to $\ddot{\varphi} \simeq 0$. Making $\ddot{\varphi} \simeq 0$ implies that $\dot{\varphi}$ is approximately constant. This gives the false vacuum plateau in Fig. 6.1.

In the sense of the rolling ball this approximation is easy understandable. The assumption of slow roll makes the acceleration $\ddot{\varphi}$ very small. This is generally true when V' and V'' are much smaller than V. That the derivatives of the potential are very small

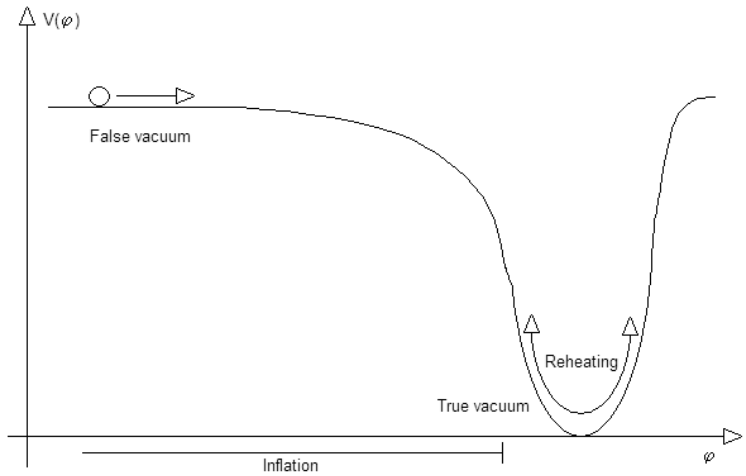

Figure 6.1: A typical Slow Roll potential. The ball on top of the false vacuum illustrates the evolution of inflation. As it is slowly rolling towards the true vacuum inflation takes place. As the ball falls down into the true vacuum inflation ends, and oscillations, corresponding to the epoch of reheating, take place as the final effect of the inflationary scenario.

compared to the potential itself ensures that H varies as

$$H(\varphi) = \sqrt{\frac{8\pi}{3m_{pl}^2} V\left(\varphi(t)\right)} \tag{6.1}$$

which gives the desired vacuum-dominated Universe. These assumptions make it possible to simplify the equations outlined in Sec. 4

$$\dot{\varphi} \simeq -\frac{m_{pl}^2}{4\pi} H'(\varphi) \tag{6.2}$$

$$\frac{\ddot{\mathcal{R}}}{\mathcal{R}} \simeq H^2 \left(1 - \frac{2}{3}\epsilon\right) \tag{6.3}$$

$$N \simeq \frac{2\sqrt{\pi}}{m_{pl}} \int_{\varphi_f}^{\varphi_i} \frac{1}{\sqrt{\epsilon(\varphi)}} d\varphi \tag{6.4}$$

where Eq. (6.4) follows from the approximation Eq. (6.2) and Eq. (4.16). Therefore $dt > 0 \Rightarrow dN < 0$. We note that $\sqrt{\epsilon}$ is defined to have the same sign as $H'(\varphi)$.

$$\sqrt{\epsilon} \equiv +\frac{m_{pl}}{2\sqrt{\pi}} \frac{H'}{H} \tag{6.5}$$

6.1 The Parameters ϵ, η and ξ^2

In Eq. (4.1) the quantum effects were ignored. Nevertheless, these give rise to scale-invariant fluctuations, which because of the rapid expansion, are redshifted to large wavelengths.

The spectrum of these fluctuations is, according to [Kinney, 2002], given by

$$P_R^{1/2}(\kappa) = \sqrt{\frac{\kappa^3}{1\pi^2}}\left|\frac{u_\kappa}{\beta}\right| \tag{6.6}$$

Here κ is a co-moving wave number and u_κ is a mode function satisfying the differential equation

$$\frac{d^2u_\kappa}{d\sigma^2} + \left(\kappa^2 - \frac{1}{\beta}\frac{d^2\beta}{d\sigma^2}\right)u_\kappa = 0 \tag{6.7}$$

where $d\sigma$ is the conformal time defined as $d\sigma = dt/\mathcal{R}$. The quantity β is defined as

$$\beta \equiv \frac{2\sqrt{\pi}}{m_{pl}}\left(\frac{\mathcal{R}\dot{\varphi}}{H}\right) = -\mathcal{R}\sqrt{\epsilon} \tag{6.8}$$

The quantity β arises in linear perturbation theory when dealing with the power spectra and the mode function. Watching the steps carefully the β-part in Eq. (6.7) can be derived [Kinney, 2002]

$$\frac{1}{\beta}\frac{d^2\beta}{d\sigma^2} = 2\mathcal{R}^2H^2\left(1 + \epsilon - \frac{3}{2}\eta + \epsilon^2 - 2\epsilon\eta + \frac{1}{2}\eta^2 + \frac{1}{2}\xi^2\right) \tag{6.9}$$

The parameters η and ξ^2 can be defined as in Eqs. (6.11) and (6.12) below. They are written along with ϵ from Sec. 4.

$$\epsilon \equiv \frac{m_{pl}^2}{4\pi}\left(\frac{H'(\varphi)}{H(\varphi)}\right)^2 \tag{6.10}$$

$$\eta \equiv \frac{m_{pl}^2}{4\pi}\left(\frac{H''(\varphi)}{H(\varphi)}\right) \tag{6.11}$$

$$\xi^2 \equiv \frac{m_{pl}^4}{16\pi^2}\left(\frac{H'(\varphi)H'''(\varphi)}{H^2(\varphi)}\right) \tag{6.12}$$

The quanteties ϵ, η and ξ^2 are the *SR parameters*. These parameters play a crucial role in describing the SR model of inflation.

The assumptions made to ensure SR can now be rephrased. Since $V', V'' \ll V$ and Eq. (6.1) is obtained, it can be inferred that in order to have SR $\epsilon \ll 1$ and $|\eta| \ll 1$, since the potential ratios in the equations beneath becomes small.

$$\epsilon \simeq \frac{m_{pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2 \tag{6.13}$$

$$\eta \simeq \frac{m_{pl}^2}{4\pi}\frac{1}{2}K\left(\frac{\frac{V''}{V^{1/2}} - \frac{V'}{2V^{3/2}}V'}{KV^{1/2}}\right) = \frac{m_{pl}^2}{8\pi}\left(\frac{V''}{V^{1/2}} - \frac{1}{2}\left(\frac{V'}{V}\right)^2\right) \tag{6.14}$$

When SR is valid the tensor and scalar power spectra, P_T and P_R respectively, are, according to [Kinney, 2002], power laws which give a spectral index n.

$$n = 1 - 4\epsilon + 2\eta \tag{6.15}$$

The variation of the spectral index with respect to the wave number is given by

$$\alpha = \frac{dn}{d\ln\kappa} \tag{6.16}$$

When $\alpha \neq 0$, one says that the inflationary model is *running*.
The tensor spectral index n_T is

$$n_T = -2\epsilon \tag{6.17}$$

Furthermore, the scalar to tensor ratio r is defined as [Kinney et al., 2006]

$$r \equiv \frac{P_T}{P_R} \simeq 16\epsilon \tag{6.18}$$

We thus see that the SR parameters are connected to the spectral indices and the power spectra. And since these are observables, we get that the SR can be tested against observations. We are now able to make observational evidence of inflation from for instance CMB.

7 Measuring Inflation

The parameters n, n_T and r are used when comparing inflationary models with observations. The most recent observations are the Wilkinson Microwave Anisotropy Probe 3 year data (WMAP3) [Spergel et al., 2006]. These data combined with the data from the Sloan Digital Sky Survey (SDSS) have been analyzed by [Kinney et al., 2006]. They use the data to set constraints on the potentials $V(\varphi)$ for various single field inflationary models, and to discriminate between these.

The idea is to use the measurements to find the form of the inflationary potential. The fact that observations are directly connected to the SR parameters ϵ and η through the power spectra and Eqs. (6.15) and (6.18), makes it possible to compare a given potential form to the observed data.

In [Kinney et al., 2006] the *Monte Carlo reconstruction* is used to calculate the duration and the effects of inflation for a huge number of different models, i.e., models with different n and r parameters. The Monte Carlo reconstruction is a stochastic method for dealing with observational constraints in order to create inflationary potentials. After having made the calculations, the idea is to plot the obtained data points in for instance the n vs. r space or, for running models, the α vs. r space. These calculations can be compared to real observations and to different types of inflationary potentials.

The process of making the constraints on the scalar potentials is rather simpel. As shown in [Kinney et al., 2006] it is possible to estimate n and to put a upper bound on r. Using that $r < 0.3$ and $n = 0.95$ combined with Eq. (6.15) and Eq. (6.18) gives

$$\epsilon < 0.019 \quad ; \quad \eta < 0.012 \tag{7.1}$$

These upper bounds on the SR parameters thus give an estimate of the scalar field potentials since the SR parameters, as shown in Eqs. (6.11) and (6.10), are directly connected to H, which in the inflationary scenario depends on V. Hence, the calculations give

$$\left(\frac{H'(\varphi)}{H(\varphi)}\right)^2 < \frac{4\pi}{m_{pl}^2} 0.019 \tag{7.2}$$

$$\left(\frac{H''(\varphi)}{H(\varphi)}\right) < \frac{4\pi}{m_{pl}^2} 0.012 \tag{7.3}$$

The results of [Kinney et al., 2006] are that models with even just a weak SR are strongly favored in the region of no running ($\alpha \neq 0$) with a "red" power spectrum, i.e, $n < 1.0$. This is exactly the characteristics of simple single field inflation models. Furthermore, [Kinney et al., 2006] finds that potentials of the form $V(\varphi) \propto \varphi^4$ are disfavored, whereas potentials of the form $V(\varphi) \propto \varphi^2$ are consistent with all the tested data sets. It should be mentioned that only upper bounds are made. Hence, the single field inflation models with an SR, as the ones outlined in Sects. 4 and 6, are consistent with the WMAP3 and the SDSS observations.

8 The Problems of Inflation

It is tempting to suppose that everything is in order. Unfortunately this is not the case. The inflationary cosmology gives a very successful scenario for the cosmology of the early Universe. However, the introduction of inflation using scalar fields leads to a new set of problems.

8.1 Fluctuation Problem

In Sec. 6.1 it was mentioned that inflationary cosmology produces an almost scale-invariant spectrum of cosmological fluctuations. The problem is that the predicted amplitude of the spectrum exceeds the observational data by several orders of magnitude [Brandenberger, 2005]. If the amplitude has to match the observations, the coefficient of the potential of inflation has to lie below a certain value, depending on the form of the potential. This constraint is not very satisfying, since one of the main goals of the inflationary scenario is to avoid fine-tunings of parameters of the cosmology.

8.2 Singularity Problem

The CSM has a singularity, the Big Bang, from which the Universe has evolved. The existence of this singularity is not explained by the CSM, which makes the model incomplete. By inferring the inflation mechanism some initial conditions to the CSM are determined, but the original singularity is neither excluded nor explained.

8.3 Cosmological Constant Problem

A widely discussed problem in cosmology concerns the observed smallness of the cosmological constant Λ. The problem is that during inflation the vacuum energy is the dominating energy component. This is not in agreement with the present smallness of Λ.

The Einstein field equations with a Λ-term is given by Eq. (2.5) According to experimental data

$$T'_{\mu\nu} \sim 10^8 cm^{-4} \Rightarrow 8\pi G' T'_{\mu\nu} \sim 5 \cdot 10^{-57} cm^{-2} \tag{8.1}$$

and

$$\Lambda' \sim 10^{-56} cm^{-2} \tag{8.2}$$

The primes indicates that the units is not the same as in this report. To be able to compare Λ' with the results in this report the unit s^{-2} is needed. This is accomplished by multiplication with c

$$\Lambda = \Lambda' \cdot c^2 \sim 10^{-35} cm^{-2} \tag{8.3}$$

Hence, the Λ is still very small and the problem still exists. If the Einstein field equation is the correct description of the dynamics of the Universe, the right hand side of Eq. (2.5) has to be of the same order as the experimental results. But an estimate of the present right hand side of Eq. (2.5) in the framework of quantum field theory, gives an extremely large value

$$8\pi G \langle T_{\mu\nu} \rangle_0 \sim 10^{66} cm^{-2} \tag{8.4}$$

Since the cosmological constant is not near this order of magnitude, the observationally right hand side of Eq. (2.5) does not seems to equal the theoretical predicted right hand side, which is the problem. [Dolgov, 1995]
Today, the cosmological constant problem is one of the worst fine-tuning problems in cosmology and is neither explained by inflation nor any other theory of particle physics.

8.4 String Theory - A Possible Solution

The problems with the inflationary cosmology arise from an incomplete understanding of fundamental physics at ultra-high energies. A possible theory providing a framework to resolve the problems, is the *String theory*. String theory contains many scalar fields which are massless prior to inflation. These provide a possibility to solve the fluctuation problem and in some specific cases the problem with cosmological singularities. The only problem with inflation which does not appear to be solvable within the current knowledge of string theory, is the cosmological constant problem.

8.5 Discussion

It is shown that with around 60 e-foldings the inflation model has the ability to solve the flatness, horizon, large scale and monopole problems. Furthermore, as explained in Sec. 7, recent data is in agreement with the single field inflation model. Thus, there are strong indications that inflation might be the wanted supplement to the CSM. Nevertheless, the inflationary scenario is still just a phenomenological description.

Since the SR provides us with cosmological observations, some indications are given that inflation actually is a part of the description of the early Universe. However, it is clear that the final statement of the evolution of the Universe cannot be given before new ideas, of how to make furhter observations, are proposed.

9 Conclusion

When describing the evolution of the Universe, the CSM is still the governing framework, although some discrepancies have arisen. Four problems of the CSM have been particulary conspicuous; the horizon, flatness, large scale structures and monopole problems. It is shown that these problems are solved by inferring a duration of the inflationary scenario of around 60 e-foldings. Inflation states that the Universe is increased by 27 orders of magnitude from 10^{-36} to $10^{-31}s$. This immense expansion solves:

- **The horizon problem**, since all regions in the Universe could have been causally connected before inflation. Inflation blows the entropy of a region of only $10^{-23}cm$

up to 10^{92}, and the entropy of today (10^{86}) thus easily fits inside, and the problem is avoided.

- **The flatness problem**, since any curvature of the Universe would be straightened out. We have shown that every pre-inflationary density parameter's difference from one is lowered by 54 orders of magnitude. Thereby, inflation turns an even extremely curved space into flat space.
- **The monopole problem**, since any amount of topological defects would be extremely diluted. This is true because any number density of monopoles is lowered by 78 orders of magnitude during inflation.

The problem with the inexplicable **large scale structures** is solved by inflation, since the scalar field has quantum fluctuations, which is the foundation for creation of the large scale structures. Since these are "frozen out" because of the huge expansion, they survive until the CSM sets¡ in.

To give the best description of the circumstances in the early Universe, many models within the broad framework of inflation, have been proposed. Investigations show that in general simple single field models of inflation are favored; and in particular, the SR model which provides a set of parameters which can be compared with observations.
Hence, the conclusion is that the inflationary SR is favored by observations and is therefore an acceptable solution to the problems concerning the initial conditions of the CSM.

10 Summary in Danish

I rapporten "The Inflationary Scenario in The Early Universe" er der givet en introduktion til den Kosmologiske Standard Big Bang Model (CSM). Herunder er Friedmann-ligningen udledt ved hjælp af simpel tensorregning. CSM understøttes af astronomiske observationer, dog har modellen nogle alvorlige forklaringsproblemer, horisont-, fladheds-, storskalastruktur- og monopolproblemet. Disse problemer har at gøre med begyndelsebetingelserne for CSM. **Horisontproblemet** opstår, idet der i dag observeres en kosmologisk mikrobølge baggrundsstråling (CMB) med en temperatur $T = 2.73K$ og med temperaturfluktuationer $\frac{\Delta T}{T} = 10^{-5}$. Da CSM ikke gør det muligt for alle punkter på himlen at være kausalt forbundne, er det et problem at $T_{CMB,0}$ er den samme over hele himlen med så stor nøjagtighed. **Fladhedsproblemet** opstår da observationer i dag viser, at densitetsparameteren $\Omega = 1 \pm 0.2$. Da $\Omega = \frac{\rho(t)}{\rho_c(t)}$ og $\rho_c(t)$ er den kritiske energidensitet, der gør Universet fladt, betyder det at der er stærk evidens for at Universet tidligere har været meget fladt. Det reelle problem opstår, da beregninger viser at Universet, da det var et sekund gammelt, var fladt med en nøjagtighed på $6 \cdot 10^{-16}$. Denne nøjagtighed kan ikke forklares af CSM. Eksistensen af **storskalastrukturer** i Universet er en kendsgerning, men ifølge det *kosmologiske princip* er Universet homogent og isotropt. Problemet opstår, idet der observeres storskalastrukturer men at eksistensen af disse under udledningen af CSM udelukkes. Hvis CSM skal gælde for hele Universets udvikling, må der derfor have været en mekanisme, der på et tidspunkt har brudt denne homogenitet og isotropi. Det sidste problem, **monopolproblemet**, opstår idet teoretiske beregninger viser, at der i det meget tidlige Univers, var en høj koncentration af magnetiske monopoler mv. Da disse endnu ikke er observeret, er det et problem at de skulle have domineret det tidlige Univers. En accepteret løsning til CSM-problemerne er inflationsscenariet. Under inflationen gennemgik Universet en ekstrem eksponentiel udvidelse. Denne forårsagedes af et skalarfelt og det tilhørende potential. De generelle ligninger for inflationen er beskrevet, hvorefter disse er brugt til at løse de fire CSM-problemer. Til løsningen af problemerne benyttes en varighed af inflationen svarende til 60 Hubble tider.
Udregninger giver, at horisontafstanden d_H under inflationen vokser med en faktor 10^{27}, hvormed de kausalt forbundne regioner blæses op. Derved kan disse let rumme det observerbare Univers, som vi ser i dag. Desuden vises det, at $|\Omega - 1|$ før inflationen bliver sænket med 54 størrelsesordener. Universet bliver dermed under inflationen tvunget til at være ekstremt fladt. Der argumenteres for at storskalastrukturerne er opstået fra det kvantemekaniske bidrag til skalarfeltet. Sådanne kvanteeffekter vil på grund af den ekstreme ekspansion af Universet blive "frosset ude" og derved omdannes til klassiske densitetsfluktuationer. Disse densitetsfluktuationer forårsagder et gravitationelt indfald, som danner storskalastrukturerne. Til sidst vises det, at en given koncentration af monopoler bliver formindsket med en faktor 10^{78} og dermed er der så godt som ingen tilbage at observere i dag.

Yderligere indføres *Slow Roll approksimationen* (SR), der er en approksimation til de generelle inflationsligninger. SR forudsiger at skalarfeltet φ foretog en langsomt rullende overgang fra falsk til ægte vakuum under inflationen. I forbindelse med udarbejdelse af modellen angives et sæt af *Slow Roll parametre*, der kan benyttes til at sammenligne teori med observationer. Selvom inflationen løser CSM-problemerne, opstår der et nyt set af problemer. Disse er kort beskrevet og diskuteret. Konklusionen på rapporten er, at SR inflation med en varighed på ~ 60 Hubble tider er et acceptabelt supplement til CSM.

References

[Bergstrom and Goobar, 1999] Bergstrom, L. and Goobar, A. (1999). *Cosmology and Particle Astrophysics (Wiley-Praxis Series in Astronomy & Astrophysics)*. Praxis Publishing.

[Brandenberger, 2005] Brandenberger, R. H. (2005). Challenges for inflation cosmology. *Physical Letters*, arXiv:astro-ph/0411671.

[Dolgov, 1995] Dolgov, A. D. (1995). Problems of Dark Matter. arXiv:astro-ph/9509064.

[Easther and Kinney, 2003] Easther, R. and Kinney, W. H. (2003). Monte Carlo reconstruction of the inflationary potential. *Physical Review D*, 67(4):043511–+, arXiv:astro-ph/0210345.

[Guth, 1981] Guth, A. H. (1981). Inflationary universe: A possible solution to the horizon and flatness problems. *Physical Review D*, 23:347–356.

[Hansen and Kunz, 2002] Hansen, S. H. and Kunz, M. (2002). Observational constraints on the inflaton potential combined with flow equations in inflaton space. *Monthly Notices of the RAS*, 336:1007–1010, arXiv:hep-ph/0109252.

[Kinney, 1998] Kinney, W. H. (1998). Constraining inflation with cosmic microwave background polarization. *Physical Review D*, 58(12):123506–+, arXiv:astro-ph/9806259.

[Kinney, 2002] Kinney, W. H. (2002). Inflation: Flow, fixed points, and observables to arbitrary order in slow roll. *Physical Review D*, 66(8):083508–+, arXiv:astro-ph/0206032.

[Kinney et al., 2006] Kinney et al., W. H. (2006). Inflation model constraints from the Wilkinson Microwave Anisotropy Probe three-year three-year data. *Fermilab-Pub-06-106-A, CERN-ph-th/2006-087*, arXiv:astro-ph/0605338.

[Kolb and Turner, 1999] Kolb, E. and Turner, M. S. (1999). *The Early Universe (Frontiers in Physics S.)*. Perseus Books.

[Lachieze-Rey et al., 1999] Lachieze-Rey, M., Gunzig, E., and Simmons, J. (1999). *The Cosmological Background Radiation*. Cambridge University Press.

[Linde, 2005] Linde, A. (2005). Particle Physics and Inflationary Cosmology. *ArXiv High Energy Physics - Theory e-prints*, arXiv:hep-th/0503203.

[Linde, 1982] Linde, A. D. (1982). A New Inflationary universe scenarion: A possible solution of the horizon, flatness, homogeneity, isortopy and primordal monopole problems. *Physical Letters*, 108(6):389–393.

[Lyth and Riotto, 1999] Lyth, D. H. and Riotto, A. A. (1999). Particle physics models of inflation and the cosmological density perturbation. *Physics Reports*, 314:1–2, arXiv:hep-ph/9807278.

[Olesen and Ambjrn, 2003] Olesen, P. and Ambjrn, J. (2003). *General Relativity Cosmology and Classical Gauge Theories (Lecture notes)*. University of Copenhagen, The Niels Bohr Institute.

[Ryden, 2003] Ryden, B. (2003). *Introduction to cosmology.* Introduction to cosmology / Barbara Ryden. San Francisco, CA, USA: Addison Wesley, ISBN 0-8053-8912-1, 2003, IX + 244 pp.

[Spergel et al., 2006] Spergel, D. N., Bean, R., Dore', O., Nolta, M. R., Bennett, C. L., Hinshaw, G., Jarosik, N., Komatsu, E., Page, L., Peiris, H. V., Verde, L., Barnes, C., Halpern, M., Hill, R. S., Kogut, A., Limon, M., Meyer, S. S., Odegard, N., Tucker, G. S., Weiland, J. L., Wollack, E., and Wright, E. L. (2006). Wilkinson Microwave Anisotropy Probe (WMAP) Three Year Results: Implications for Cosmology. *ArXiv Astrophysics e-prints*, arXiv:astro-ph/0603449.

[Tsujikawa, 2003] Tsujikawa, S. (2003). Introductory review of cosmic inflation. *ArXiv High Energy Physics - Phenomenology e-prints*, arXiv:hep-ph/0304257.

[Weinberg, 1972] Weinberg, S. (1972). *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity, by Steven Weinberg, pp. 688. ISBN 0-471-92567-5. Wiley-VCH.

[Woan, 2000] Woan, G. (2000). *The Cambridge Handbook of Physics Formulas.* The Cambridge Handbook of Physics Formulas, by Graham Woan, pp. 228. ISBN 0521573491. Cambridge, UK: Cambridge University Press, June 2000.

A Rewriting Einstein's Field Equations

Below Einstein's field equations are rewritten to a form which is of great use when deriving the Friedmann equation Eq. (2.16).

We start off by stating Einstein's field equations in tensor form:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = -8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{A.1}$$

Now, by contracting this equation with the metric tensor $g^{\mu\nu}$, we get the following:

$$g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} g^{\mu\nu} g_{\mu\nu} \mathcal{R} = -8\pi G g^{\mu\nu} T_{\mu\nu} + \Lambda g^{\mu\nu} g_{\mu\nu} \tag{A.2}$$

$$\Downarrow$$

$$\mathcal{R} - \frac{1}{2} \delta^{\mu}_{\mu} \mathcal{R} = -8\pi G T^{\mu}_{\mu} + \Lambda \delta^{\mu}_{\mu} \tag{A.3}$$

And since δ^{γ}_{κ} equals 1 when $\kappa = \gamma$ and 0 when $\kappa \neq \gamma$, we get that Einstein summation over δ^{μ}_{μ} is 4, because we are dealing with 4 dimensions (time and 3D space). This gives that

$$\mathcal{R} - \frac{1}{2} 4\mathcal{R} = -8\pi G T^{\mu}_{\mu} + 4\Lambda \tag{A.4}$$

$$\Downarrow$$

$$\mathcal{R} = 8\pi G T^{\mu}_{\mu} - 4\Lambda \tag{A.5}$$

Substituting this into Eq. (A.1) then gives

$$R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} \left(8\pi G T^{\mu}_{\mu} - 4\Lambda\right) - 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{A.6}$$

$$= -8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\mu}_{\mu}\right) + (-2\Lambda + \Lambda) g_{\mu\nu} \tag{A.7}$$

$$= -8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\mu}_{\mu}\right) - \Lambda g_{\mu\nu} \tag{A.8}$$

Now, assuming that the energy momentum tensor $T_{\mu\nu}$ takes the form of a perfect fluid

$$T_{\mu\nu} = p g_{\mu\nu} + (p + \rho) U_{\mu} U_{\nu} \tag{A.9}$$

Einstein's field equations can be expressed as beneath. In Eq. (A.9) p and ρ are the pressure and the density respectively and the Us are four velocities in the four dimensions. One should note that p and ρ are only dependent on time!

$$R_{\mu\nu} = -8\pi G \left[p g_{\mu\nu} + (p + \rho) U_{\mu} U_{\nu} - \frac{1}{2} g_{\mu\nu} T^{\mu}_{\mu}\right] - \Lambda g_{\mu\nu} \tag{A.10}$$

$$= -8\pi G \left[\frac{1}{2} g_{\mu\nu} \left(2p - T^{\mu}_{\mu}\right) + (p + \rho) U_{\mu} U_{\nu}\right] - \Lambda g_{\mu\nu} \tag{A.11}$$

$$= -8\pi G \frac{1}{2} g_{\mu\nu} \left(2p - \left(p g^{\mu}_{\mu} + (p + \rho) U_{\mu} U^{\mu}\right)\right) -8\pi G (p + \rho) U_{\mu} U_{\nu} - \Lambda g_{\mu\nu} \tag{A.12}$$

$$= -8\pi G \frac{1}{2} g_{\mu\nu} \left(2p - \left(p \delta^{\mu}_{\mu} + (p + \rho) g_{\mu\mu} U^{\mu} U^{\mu}\right)\right) -8\pi G (p + \rho) U_{\mu} U_{\nu} - \Lambda g_{\mu\nu} \tag{A.13}$$

Using the values stated in Eqs. (2.7) and (2.10) in section 2.2 and using that $g_{\mu\mu}U^\mu U^\mu$ is -1 for $\mu = t$ and 0 for $\mu = i$ which using Einstein summation over μ gives that $g_{\mu\mu}U^\mu U^\mu$ is in total -1, we get the following:

$$\begin{aligned} R_{\mu\nu} &= -8\pi G\frac{1}{2}g_{\mu\nu}\left(2p - (4p - (p+\rho))\right) \\ &\quad -8\pi G(p+\rho)U_\mu U_\nu - \Lambda g_{\mu\nu} \\ &= -8\pi G\left[\frac{1}{2}(\rho - p)g_{\mu\nu} + (p+\rho)U_\mu U_\nu\right] - \Lambda g_{\mu\nu} \end{aligned}$$

$$\Downarrow$$

$$R_{\mu\nu} = -8\pi G\left[\frac{1}{2}(\rho - p)g_{\mu\nu} + (p+\rho)g_{\mu\nu}U^\nu g_{\mu\nu}U^\mu\right] - \Lambda g_{\mu\nu} \tag{A.14}$$

Eq. (A.14) is the desired form of Einstein's field equations used in the derivation of the Friedmann equation in Sec. 2.2, obtained only by tensor manipulation, assuming the perfect fluid form of the energy momentum tensor and the definitions of the four-velocities and the $g_{\mu\mu}$s.

B Deriving the Equation of Motion

The starting point when deriving the equation of motion is the energy momentum tensor of the form of a perfect fluid

$$T^{\mu\nu} = -pg^{\mu\nu} + (p+\rho)U^\mu U^\nu \tag{B.1}$$

Furthermore, we know from [Bergstrom and Goobar, 1999] that a Lagrangian on the form

$$\mathcal{L} = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - V\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi) \tag{B.2}$$

which is "useful for cosmological applications", gives a contribution to $T^{\mu\nu}$ so that

$$T^{\mu\nu} = \partial^\mu\varphi\partial^\nu\varphi - \mathcal{L}g^{\mu\nu} \tag{B.3}$$

where $\frac{\partial}{\partial x^\mu}$ for simplicity is written as ∂^μ.

It is assumed that the scalar field φ is spatially homogeneous, thus $\partial^i\varphi = 0$. This assumption seems fair since in the CSM we have already assumed that the Universe in general is isotropic and homogeneous. Combining Eqs. (B.1) and (B.3) gives that

$$T^{\mu\nu} = \begin{cases} \partial^t\varphi\partial^t\varphi - \left(\frac{1}{2}\dot{\varphi}^2 - V(\varphi)\right)g^{tt} \\ \partial^i\varphi\partial^j\varphi - \left(\frac{1}{2}\dot{\varphi}^2 - V(\varphi)\right)g^{ij} \end{cases} \tag{B.4}$$

According to the Einstein summation and basic tensor calculus [Weinberg, 1972], $g^{tt}g_{tt} = \delta^t_t = -1$, and since g_{tt} according to Eq. (2.7) is -1, we have that $g^{tt} = 1$. From (2.7) we also have that $g_{ij} = g^{ij} = 0$ for $i \neq j$. This gives that the non-vanishing parts of the energy momentum tensor are given by

$$T^{tt} = \dot{\varphi}^2 - \left(\frac{1}{2}\dot{\varphi} - V(\varphi)\right) = \frac{1}{2}\dot{\varphi} + V(\varphi) \tag{B.5}$$

$$T^{ii} = -g^{ii}\left(\frac{1}{2}\dot{\varphi} - V(\varphi)\right) \tag{B.6}$$

Eq. (B.1) together with the definitions Eqs. (2.10) and (2.7) give that the only non-vanishing components of the energy momentum tensor could also be written as

$$T^{tt} = -pg^{tt} + (p+\rho)U^tU^t = \rho \tag{B.7}$$
$$T^{ii} = -pg^{ii} + (p+\rho)U^iU^i = -pg^{ii} \tag{B.8}$$

If this is combined with Eqs. (B.5) and (B.6), the energy density and the pressure can be expressed as a function of the inflaton φ

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \quad , \quad p_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi) \tag{B.9}$$

According to [Weinberg, 1972] the conservation of the energy momentum tensor ($T^{\mu\nu}{}_{;\nu} = 0$) can be written as

$$T^{\mu\nu}{}_{;\mu} = \frac{1}{\sqrt{g}}\partial^\mu\left(\sqrt{g}T^{\mu\nu}\right) + \Gamma^\nu_{\mu\lambda}T^{\mu\lambda} = 0 \tag{B.10}$$

where $g = -Det(g_{\mu\nu})$ equals $\mathcal{R}^6$ for the RW metric.

The quantity $\Gamma^\lambda_{\mu\nu}$ in Eq. (B.10) is called the *affine connection.* In the tensor formalism the equation of motion of a particle under influence of the gravitational forces is defined as

$$\Gamma^\lambda_{\mu\nu} \equiv \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \tag{B.11}$$

where the ξ^αs are the coordinates in the freely falling coordinate system where the particle is located. The x represents any other coordinate system (for instance Cartesian coordinates). The affine connection is itself not a tensor but is connected to the Ricci tensor via the equation:

$$R^\lambda_{\mu\lambda\nu} \equiv R_{\mu\nu} = \frac{\partial \Gamma^\lambda_{\lambda\mu}}{\partial x^\nu} - \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} + \Gamma^\eta_{\lambda\nu}\Gamma^\lambda_{\eta\mu} - \Gamma^\lambda_{\mu\nu}\Gamma^\eta_{\lambda\eta} \tag{B.12}$$

It can be shown that some of the elements of the affine connection in the RW case equal zero. This implies that only some of the elements of the Ricci tensor in the RW metric case are non-vanishing. The non-vanishing parts of the affine connection are [Weinberg, 1972]

$$\begin{aligned} \Gamma^t_{ij} &= \mathcal{R}\dot{\mathcal{R}}\tilde{g}_{ij} \\ \Gamma^i_{tj} &= \tfrac{\dot{\mathcal{R}}}{\mathcal{R}}\delta^i_j \;=\; H\delta^i_j \\ \Gamma^i_{jk} &= \tfrac{1}{2}(\tilde{g}^{-1})^{il}\left(\partial^k\tilde{g}_{lj} + \partial^j\tilde{g}_{lk} + \partial^l\tilde{g}_{jk}\right) \end{aligned} \tag{B.13}$$

Combining Eq. (B.10) and the nonvanishing parts of the affine connection with the expressions for the energy momentum tensor components, gives some relations between the density, the pressure and their variations in space and time. In particular using the time components gives the fluid equation known from Sec. (2.2). Hence, combining the results above gives

$$0 = \dot{\rho} + 3H(\rho + p) \tag{B.14}$$

Using this and the expressions (B.9) for ρ and p finally gives the *classical equation* of motion for the inflaton scalar field φ.

$$
\begin{aligned}
0 &= \partial^t\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right) + 3H\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi) + \frac{1}{2}\dot{\varphi}^2 - V(\varphi)\right) \\
&\Downarrow \\
0 &= \ddot{\varphi} + 3H\dot{\varphi} + \frac{\partial^t V(\varphi)}{\partial^t \varphi} \\
&\Downarrow \\
0 &= \ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) \qquad \text{(B.15)}
\end{aligned}
$$

Here an dot represents derivation with respect to time and a prime is derivation with respect to φ

C Calculations Leading to Expression for $\mathcal{R}(t)$

Firstly, a couple of assumptions are made:

- **A flat Universe:** Gives that $k = 0$. This is reasonable because, as mentioned in Sec. 3.2, calculations show that $\Omega - 1$ has been very close to zero for almost the entire life of the Universe. Today it is believed that $0.8 < \Omega < 1.2$ [Ryden, 2003].
- **Specific equation of state:** $p = w\rho$. According to [Ryden, 2003] this is a reasonable assumptions when dealing with cosmology. This ensures that we can use the expression (2.25) for ρ from Sec. 2.3.
- **Single component Universe:** That is ρ is dominated by only one component so that $\rho = \rho_{w,0}\mathcal{R}^{-3(1+w)}$. This is also reasonable as long as we only use the equations for either matter, radiation or cosmological constant dominated eras.
- **Expanding Universe:** This gives that $\dot{\mathcal{R}} > 0$.

Using these assumptions and the Friedmann equation Eq. (2.16) gives the following:

$$
\begin{aligned}
\frac{\dot{\mathcal{R}}^2}{\mathcal{R}^2} &= \frac{8\pi G}{3}\rho \qquad \text{(C.1)} \\
&\Downarrow \\
\frac{d}{dt}\mathcal{R} &= \left(\frac{8\pi G}{3}\rho_{w,0}\mathcal{R}^{-3(1+w)}\mathcal{R}^2\right)^{1/2} \qquad \text{(C.2)} \\
&\Downarrow \\
\mathcal{R}^{\frac{1+3w}{2}}\frac{d}{dt}\mathcal{R} &= \left(\frac{8\pi G}{3}\rho_{w,0}\right)^{1/2} \qquad \text{(C.3)}
\end{aligned}
$$

In the step from Eq. (C.1) to Eq. (C.2) we chose $\sqrt{\left(\frac{d}{dt}\mathcal{R}\right)^2}$ to be positive to obey the fourth assumption.
Now defining the constant $C = \left(\frac{8\pi G}{3}\rho_{w,0}\right)^{1/2}$ and using that

$$
\mathcal{R}^{\frac{1+3w}{2}}\frac{d}{dt}\mathcal{R} = \frac{d}{dt}\left(\left(\frac{1+3w}{2}+1\right)^{-1}\mathcal{R}^{\frac{1+3w}{2}+1}\right)
$$

gives the following.

$$\frac{d}{dt}\left(\mathcal{R}^{\frac{1+3w}{2}+1}\right) = \left(\frac{1+3w}{2}+1\right)C \tag{C.4}$$

$$\Downarrow$$

$$\int d\left(\mathcal{R}^{\frac{3(1+w)}{2}}\right) = \int\left(\frac{3(1+w)}{2}C\right)dt \tag{C.5}$$

$$\Downarrow$$

$$\mathcal{R}^{\frac{3(1+w)}{2}} = \left(\frac{3(1+w)}{2}C\right)t \tag{C.6}$$

$$\Downarrow$$

$$\mathcal{R}(t) = \left(\left(\frac{3(1+w)}{2}C\right)t\right)^{\frac{2}{3(1+w)}} \tag{C.7}$$

Calculating the dimension (and multiplying with c^2 to correct for natural units), we see that the constant $\frac{3(1+w)}{2}C$ has dimension s^{-1}. This normalizes the time, so that $\mathcal{R}(t)$ is dimensionless as expected, and defining the constant as $\frac{1}{t_0}$ we have the desired expression

$$\mathcal{R}(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} \tag{C.8}$$

D The Expression for the Total Entropy S

To be able to define the entropy density s, we need an expression for the total entropy. In doing so we rewrite the first law of thermodynamics in the differential form and integrate the expression for dS.
The first law of thermodynamics reads

$$dS = \frac{d(\rho V) + pdV}{T} = \frac{d\left((\rho+p)V\right) - V dp}{T} \tag{D.1}$$

Manipulating the differentials, gives

$$dS = \frac{1}{T}\left(d(\rho V) + pdV\right) \tag{D.2}$$

$$= \frac{1}{T}\left(V d\rho + \rho dV + pdV\right) \tag{D.3}$$

$$= \frac{1}{T}(\rho+p)dV + \frac{1}{T}V[d(\rho+p) - dp] \tag{D.4}$$

$$= \frac{1}{T}d((\rho+p)V) - \frac{V}{T}dp \tag{D.5}$$

Since the entropy only depends on the temperature and the volume, it would be preferable to express the entropy as the sum of these differentials. Thus defining

$$dS = \frac{\partial S}{\partial V}dV + \frac{\partial S}{\partial T}dT \tag{D.6}$$

gives that

$$dS = \frac{1}{T}(\rho+p)dV + \frac{V}{T}\frac{d\rho}{dT}dT \tag{D.7}$$

The order of differentiation is insignificant, for the entropy and both ρ and p only depends on the temperature (meaning that $\partial p/\partial T = dp/dT$ and $\partial\rho/\partial T = d\rho/dT$). This gives that

$$
\begin{aligned}
\frac{\partial^2 S}{\partial T \partial V} &= \frac{\partial^2 S}{\partial V \partial T} \\
&\Downarrow \\
\frac{\partial}{\partial T}\left(\frac{1}{T}(\rho+p)\right) &= \frac{\partial}{\partial V}\left(\frac{V}{T}\frac{d\rho}{dT}\right) \\
&\Downarrow \\
-\frac{\rho+p}{T^2} + \frac{1}{T}\frac{\partial}{\partial T}(\rho+p) &= \frac{1}{T}\frac{d\rho}{dT} + V\left(\frac{\partial}{\partial V}\left(\frac{1}{T}\frac{d\rho}{dT}\right)\right) \\
&\Downarrow \\
\frac{dp}{dT} &= \frac{\rho+p}{T} \qquad \text{(D.8)}
\end{aligned}
$$

Combining this with the expressions for dS in Eq. (D.5) gives the expression for dS which can be integrated to get the total entropy.

$$
dS = \frac{1}{T}d((\rho+p)V) - \frac{V}{T^2}(\rho+p)dT \qquad \text{(D.9)}
$$

Integrating Eq. (D.9) and ignoring the integration constant gives

$$
\begin{aligned}
S_{tot} = \int dS &= \int \frac{1}{T}d((\rho+p)V) - \int \frac{V}{T^2}(\rho+p)dT \\
&= \left[V(\rho+p)\frac{1}{T}\right] - \int -\frac{V}{T^2}(\rho+p)dT - \int \frac{V}{T^2}(\rho+p)dT \\
&= \frac{V}{T}(\rho+p) \qquad \text{(D.10)}
\end{aligned}
$$

This is the desired expression for the total entropy density, which makes it possible to define the entropy density $s = S/V$ as

$$
s = \frac{S}{V} = \frac{\rho+p}{T} \qquad \text{(D.11)}
$$